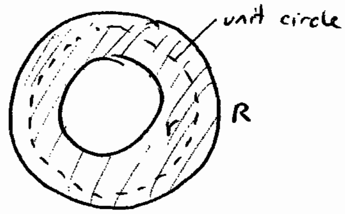
We will spend a lot of time in the unit disk – same as a compact, simply-connected subset of $\mathbb{C}$.

Laurent series

$$\sum_{n \in \mathbb{Z}} a_n z^n \qquad r < |z| < R$$



assume $r < 1 < R$

$$f(e^{it}) = \sum_{n \in \mathbb{Z}} a_n e^{int}$$

$$\sum_{n=0}^{\infty} a_n z^n$$

ρ = radius of convergence of a power series

for all z, $|z| < \rho$, the series converges

for all z, $|z| > \rho$, " diverges

ρ can be 0 or ∞, but we will assume it is finite and nonzero.

$|z| = \rho$?? that is where all the fun begins.

If $\sum |a_n|^2 < \infty$ then $\sum a_n z^n$ converges almost everywhere on $|z| = 1$.

$\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$ unit disk

Schwarz Lemma

Let $f \in \mathrm{Hol}(\mathbb{D})$ such that $|f(z)| \leq 1$ for all $z \in \mathbb{D}$, and $f(0) = 0$.

Then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

Moreover, if for some $z_0 \in \mathbb{D}$, $|f(z_0)| = |z_0|$, or if $|f'(0)| = 1$ then $f(z) = \alpha z$ with $|\alpha| = 1$.

Proof

Consider $g(z) = \dfrac{f(z)}{z}$. Then $|g(z)| \leq 1$ for $|z| = 1$.

$\Rightarrow |g(z)| \leq 1$.
by maximum modulus principle

What is wrong with this proof? Must assume function extends continuously to the boundary, so as stated, this proof does not work.

To fix proof: $g(z) \le \frac{1}{r}$ on $|z| = r < 1$.

Here, if $|z| = r$, everything is well-defined, so we can apply the maximum modulus principle.

$\Rightarrow |g(z)| \le \frac{1}{r}$ for all $|z| \le r$.

Or we can rephrase it: for all $z \in \mathbb{D}$, for all $\underset{r<1}{r \ge |z|}$, $|g(z)| \le \frac{1}{r}$

$\Rightarrow |g(z)| \le \lim_{r \to 1^-} \frac{1}{r} = 1$. So, the first statement is proved.

To prove the second statement:

$$f'(0) = \lim_{z \to 0} \frac{f(z) - \cancelto{0}{f(0)}}{z - \cancel{0}} = \lim_{z \to 0} \frac{f(z)}{z} = \lim_{z \to 0} g(z) = g(0).$$ we know $|g(0)| \le 1$.

Remember that f is analytic on $\mathbb{D}$, so it can be represented as

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = f'(0) z + \frac{f''(0)}{2} z^2 + \dots$$ so g is defined everywhere on $\mathbb{D}$, including 0.

If $|f(z_0)| = |z_0|$, then $|g(z_0)| = 1$, so $|g|$ has a local maximum at z_0
$\Rightarrow g(z) \equiv \text{constant}$

If $|f'(0)| = 1$, then $|g(0)| = 1$, so $|g|$ has a maximum at $0 \Rightarrow g(z) \equiv$ constant.

<u>Möbius transforms</u>

$\varphi: \mathbb{D} \to \mathbb{D}$. If $\varphi(z_0) = 0$, then $\varphi(z) = \alpha \frac{z - z_0}{1 - \bar{z}_0 z}$, $|\alpha| = 1$.

<u>Remark</u> $(1 - |z|^2)|f'(z)|$ is conformally invariant in $\mathbb{D}$.

$\varphi(z) = \alpha \frac{z - z_0}{1 - \bar{z}_0 z}$ then $|\varphi'(z)| = \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z|^2}$ (do this computation on your own)

and $$1 - |\varphi(z)|^2 = \frac{(1 - |z|)^2 (1 - |z_0|)^2}{|1 - \bar{z}_0 z|^2} = (1 - |z|^2) \cdot |\varphi'(z)|.$$

<u>change of variables</u>

$w = \varphi(z)$, $f(z) = g(\varphi(z)) = g(w)$

$(1 - |z|^2)|f'(z)| = (1 - |w|^2)|g'(w)|$

<u>Proof</u>

$$(1 - |w|^2)|g'(w)| = (1 - |\varphi(z)|^2)|g'(\varphi(z))| = (1 - |z|^2)|g(\varphi(z))\varphi'(z)| = (1 - |z|^2)|f'(z)|.$$

Corollary If we know that $|f| \le 1$ in $\mathbb{D}$ and $f(0)=0$,

Then $|f'(0)| \le 1$.

If $|f| \le 1$ in $\mathbb{D}$ and $f(z_0)=0$, then $f'(z_0) = ?$

Well, $(1 - |z_0|^2)\,|f'(z_0)| \le 1$

$$|f'(z_0)| \le \frac{1}{1-|z_0|^2}.$$