Conformally Invariant Schwartz Lemma

If $f \in H^\infty$, $|f(z)| \le 1$ $\forall z$, $f(z_0) = 0$, then $|f(z)| \le \left|\frac{z-z_0}{1-\bar{z}_0 z}\right|$

(change of variable, $w = \frac{z-z_0}{1-\bar{z}_0 z}$)

If $f \in H^\infty$, $|f(z)| \le 1$ $\forall z$, then (1) $\left|\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}\right| \le \left|\frac{z-z_0}{1-\bar{z}_0 z}\right|$

(2) $$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}$$

Pf| (1) is previous lemma applied to $\frac{f(z)-f(z_0)}{1-\overline{f(z_0)}f(z)}$

(2): divide (1) by $|z-z_0|$ and take $\lim_{z \to z_0}$. We get 2 at $z=z_0$

Cor: If $f \in H^\infty$, $\|f\|_\infty \le 1$ ($|f(z)| \le 1$ $\forall z$), then $|f'(z)| \le \frac{1}{1-|z|^2}$

Hyperbolic and Pseudohyperbolic Distance

$\rho(z,w) = \left|\frac{z-w}{1-\bar{w}z}\right|$ pseudohyperbolic

hyperbolic - conformally invariant metric

$(ds)^2 = Q(dx,dy)$, γ path

length$(\gamma) = \int_\gamma ds$

$Q_z(dx,dy)$

$Q_0(dx,dy) = a^2((dx)^2+(dy)^2)$

$(ds)^2 = a^2|dz|^2$

$w = \varphi_{z_0}(z) := \frac{z-z_0}{1-\bar{z}_0 z}$

$ds = a|dw|$ at $w=0$

$= a|\varphi_{z_0}'(z_0)|dz$ at $z=z_0$

$= \frac{a}{1-|z_0|^2}dz$

So, $Q_z(dz) = \frac{a^2|dz|^2}{(1-|z_0|^2)^2}$

$a=2$, $ds = \frac{2}{1-|z|^2}|dz|$

Why $a=2$? $(1-|z|^2) = (1+|z|)(1-|z|)$

$(1+|z|)$: near 2 when z near boundary

$(1-|z|)$: distance to boundary

Hyp. distance $\psi(z_1,z_2) = \ln\frac{1+\rho}{1-\rho}$, $\rho = \rho(z_1,z_2)$, pseudohyperbolic distance

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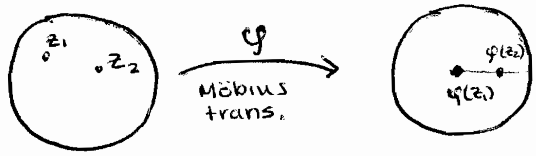


$\varphi(z_1) = 0$

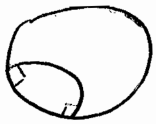
$\varphi(z_2)$ on segment connecting 0 and 1

$\rho(z_1, z_2) = \varphi(z_2)$

$$\Psi(z_1, z_2) = 2\int_0^{\rho} \frac{dx}{1-x^2} = \ln\frac{1+\rho}{1-\rho}$$

Geodesics

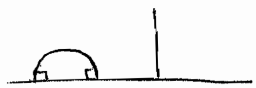
orthogonal to boundary

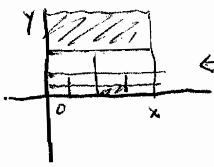


Hyperbolic metric in $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$

$$ds = \frac{|dz|}{\operatorname{Im} z}$$

Geodesics are vertical lines and circles orthogonal to boundary





← any two of these boxes are the same under the hyperbolic metric

Poisson Kernel for $\mathbb{D}$

Given $f \in C(\mathbb{T})$, $\mathbb{T} = \partial\mathbb{D}$, find $F \in \operatorname{Harm}\mathbb{D} \cap C(\overline{\mathbb{D}})$ s.t. $F|_{\mathbb{T}} = f$

$$F(z) = \int_0^{2\pi} \frac{1-|z|^2}{|1-\bar{z}e^{it}|^2} f(e^{it})\frac{dt}{2\pi} = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\xi|^2} f(\xi)\frac{|d\xi|}{2\pi},$$ where $\xi = e^{it}$, $|d\xi| = dt$

$$F(0) = \int_{\mathbb{T}} f(z)\frac{|dz|}{2\pi}$$

$G = F \circ \varphi_{z_0}^{-1}$, $\varphi_{z_0}(z) = \dfrac{z - z_0}{1 - \bar{z}_0 z}$

G is harmonic $\Leftrightarrow$ F is harmonic

$$F(z_0) = G(0) = \int_{\mathbb{T}} g(w)\frac{|dw|}{2\pi},$$ where $g = f \circ \varphi_{z_0}^{-1}$, $w = \varphi_{z_0}(z)$

$w = \varphi_{z_0}(z)$,
$$= \int_{\mathbb{T}} g(\varphi_{z_0}(z))\,|\varphi_{z_0}'(z)|\,\frac{|dz|}{2\pi}$$
$$= \int_{\mathbb{T}} f(z)\frac{1-|z_0|^2}{|1-\bar{z}_0 z|^2}\frac{|dz|}{2\pi}$$