Younghun Hong

Hardy space H^p.

- $H^p = \{ f \in Hol(\mathbb{D}) : \underbrace{\sup_{0<r<1} \int_{\mathbb{T}} |f(rz)|^p \frac{|dz|}{2\pi}}_{\substack{\| \\ \|f\|_{H^p}^p}} < \infty \}$.

H^p is defined when $0<p\le\infty$. But, it is Banach space when $1\le p\le\infty$.

Alternative charaterization of H^p.

- $f \in L^1(\mathbb{T})$. $\hat{f}(k) = \int_{\mathbb{T}} f(z)\bar{z}^k \frac{|dz|}{2\pi} = \int_0^{2\pi} f(e^{it}) e^{-ikt} \frac{dt}{2\pi}$.

$f(e^{it}) \sim \sum \hat{f}(k) e^{ikt}$.

$S_n f = \sum_{k=-n}^{n} \hat{f}(k) z^k$. If $f \in L^1$, $S_n f \not\to f$ in L^1.

$\sigma_n f = \frac{1}{n+1}(S_0 f + \cdots + S_n f)$ " , $\|\sigma_n f - f\|_1 \to 0$.

Fejer means.

Def. $\mathcal{H}^p = \{ f \in L^p(\mathbb{T}) : \hat{f}(k) = 0 \ \forall k<0 \}$.

Thm. $\mathcal{H}^p$ "=" H^p.

$H^p \ni \sum_{k\ge 0} a_k z^k \quad |z|<1$

$\mathcal{H}^p \ni \sum_{k\ge 0} b_k z^k \quad |z|=1$.

different, but there is a bijection preserving norms.

Extending harmonically $f \in L^1(\mathbb{T})$ to $\mathbb{D}$

$f \in L^1(\mathbb{T})$. $f(e^{it}) \sim \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikt}$.

$F(z) = \sum \hat{f}(k) z^k \quad |z|<1$. $\because$ it converges on $|z|<1$, because $\hat{f}(k)$ is bounded. ($f \in L^1(\mathbb{T})$.)

$= \sum_{k\ge 0} \hat{f}(k) z^k + \sum_{k>0} \hat{f}(-k) \bar{z}^k$.

$z = re^{i\theta}$. $\Rightarrow F(re^{i\theta}) = \sum_{k\in\mathbb{Z}} \hat{f}(k) r^{|k|} e^{ik\theta} = \sum_{k\in\mathbb{Z}} \left(\int_0^{2\pi} f(e^{it}) \overset{r^{|k|}}{\vee} e^{-ikt} \frac{dt}{2\pi} \right) e^{ik\theta}$

$= \int_0^{2\pi} f(e^{it}) \left[\sum_{k\in\mathbb{Z}} r^{|k|} e^{ik(\theta-t)} \right] \frac{dt}{2\pi}$.

Dominated Convergence, or Fabini's thm.
(If Both sides are finite, then we can change their order)

$$= \int_0^{2\pi} f(e^{it}) \frac{1-r^2}{|1-re^{i(\theta-t)}|^2} \frac{dt}{2\pi}$$

$$= \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\xi|^2} f(\xi) \frac{|d\xi|}{2\pi}, \qquad z = re^{i\theta},\ \xi = e^{it}.$$

Poisson integral formula.

$$P_r(e^{ix}) = \frac{1-r^2}{|1-re^{ix}|^2}.$$

$$\Rightarrow F(re^{i\theta}) = P_r * f(e^{i\theta}). \qquad \left(f*g(e^{is}) = \int_0^{2\pi} f(e^{it}) g(e^{i(s-t)}) \frac{dt}{2\pi}. \right)$$

Facts from real analysis.

1. Convolution theorem:

$$\|f*g\|_p \leq \|f\|_p \|g\|_1$$

2. $\frac{\varepsilon}{3}$ theorem:

$T_n : X \to Y$, X_0: dense subset of X.

a) $\|T_n\| \leq C < \infty$

b) $T_n x \to Tx \quad \forall x \in X_0$.

Then, $T_n x \to Tx \ \forall x \in X$. $\|T_n x - Tx\| \to 0$.

3. Banach-Alouglu theorem:

Closed unit ball in X^* is compact in W^* topology.

(X. X^*.

If $f_n \in X^*$, $\|f_n\| \leq C < \infty$, then $\exists f_{n_k}$ s.t. $f_{n_k} \xrightarrow{W^*} f$, that is,

$\forall x \in X$, $\langle x, f_n \rangle \underset{n\to\infty}{\longrightarrow} \langle x, f \rangle$)

$(L^p)^* = L^q \quad \frac{1}{p} + \frac{1}{q} = 1,\ 1 \leq p < \infty,\ 1 < q \leq \infty.$

($\forall f_n \in L^q$, $1 < q \leq \infty$, s.t. $\|f_n\|_q \leq C < \infty$

$\exists f_{n_k}, f \in L^q$ s.t. $\forall g \in L^p$ $(\frac{1}{p} + \frac{1}{q} = 1)$ $\int f_n g \to \int f g$)