$f_n \in X^*$ $\quad \|f_n\| \le C < \infty$

Was last lecture

$\Rightarrow \exists\, f_{n_k}, f$ s.t. $f_{n_k}(x) \to f(x)$ $\forall x$ if X separable, which is the case in our situation

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$\|P_r\|_1 = 1$

$\|$

$$\int_{\mathbb{T}} \frac{1-r^2}{|1-rz|^2} \frac{dz}{2\pi}$$ is harmonic ext of 1 on $\mathbb{T}$ to $r \in \mathbb{D}$

L $f \in L^p(\mathbb{T})$ $\quad 1 \le p \le \infty$

$$F(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} f(\zeta) \frac{|d\zeta|}{2\pi}$$

$f_r(\zeta) = F(r\zeta)$ then

1. $\|f_r\|_p < \|f\|$
2. for $p < \infty$ $\|f_r - f\|_p \to 0$ as $r \to 1$.

   for $p = \infty$ $\displaystyle\int_{\mathbb{T}} (f_r - f) g \frac{d\zeta}{2\pi} \to 0$ as $r \to 1$ $\quad \forall g \in L^1$

3. If $f \in \mathcal{H}^p \Rightarrow F \in H^p$, $\|F\|_p = \|f\|_p$

Pf 1. follows from convolution thm since $f_r = P_r * f$.

2. Use $\frac{\varepsilon}{3}$ thm. Trivial for trig polynomials $f = \sum_{-N}^{N} \hat{f}(k) \zeta^k$

$\zeta \in \mathbb{T}$ $\quad f_r = \sum_{-N}^{N} \hat{f}(k) r^{|k|} \zeta^k$

Trig poly are dense in L^p $1 \le p < \infty$,

so by $\frac{\varepsilon}{3}$ thm can extend $f_r \to f$ to L^p. (skip case of ∞.)

3. $f \in \mathcal{H}^p$ $\hat{f}(k) = 0$ for $k < 0$.

Then $F(z) = \sum_{k \geq 0} \hat{f}(k) z^k \in Hol(\mathbb{D})$ and

$$\int_{\mathbb{T}} |F(r\zeta)|^p \frac{d\zeta}{2\pi} \leq \|f\|_p^p$$

so $\|F\|_{H^p} \leq \|f\|_p$.

Rem: $\forall F \in Hol(\mathbb{D})$ $\int_{\mathbb{T}} |F(r\zeta)|^p \frac{|d\zeta|}{2\pi} \nearrow$ as $r \nearrow 1$.

$f_r(z) = F(rz)$ $z \in \mathbb{T}$

If $r_1 < r_2$ then $f_{r_1} = P_{r_1/r_2} * f_{r_2}$.

So in def of H^p, $\sup_r$ can be replaced by $\lim_{r \to 1}$.

So in the definition of H^p

$\mathcal{H}^p \subset H^p$ proved.

so $\|F\|_{H^p} = \lim_{r \to 1-} \|f_r\|_p \underset{\nearrow}{=} \|f\|_p$

because $\|f_r - f\|_p \underset{r \to 1-}{\to} 0$

Want $H^p \subset \mathcal{H}^p$.

Case $p > 1$. $F \in H^p$. Define $f_r(\zeta) = F(r\zeta)$ $\zeta \in \mathbb{T}$

$\exists r_k \nearrow 1$ s.t. $\int_{\mathbb{T}} f_{r_k}(\zeta) g(\zeta) \frac{|d\zeta|}{2\pi} \longrightarrow \int f(\zeta) g(\zeta) \frac{d\zeta}{2\pi}$ $\forall g \in C(\mathbb{T})$.

$\exists f \in L^p$

If $g(\zeta) = \zeta^n$ $n > 0$ $\hat{f}(-n) = \int f(\zeta) \zeta^n \frac{|d\zeta|}{2\pi}$

$$= \lim_{k \to \infty} \underbrace{\int_{\mathbb{T}} f_{r_k}(\zeta) \zeta^n \frac{|d\zeta|}{2\pi}}_{\text{claim} \, = \, 0}$$

because $f_{r_k} \in \mathcal{H}^p$ (or by Cauchy theorem).

Using Cauchy theorem: $z = e^{it}$ $\quad dz = ie^{it}\,dt$

$$|dz| = \frac{dz}{iz}$$

$$F(z) = \lim_{k\to\infty} F(r_k z) = \lim_{k\to\infty} \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} f_{r_k}(\zeta)\,\frac{d\zeta}{2\pi}$$

$$= \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} f(\zeta)\,\frac{|d\zeta|}{2\pi}$$

What about $p=1$

$M(\mathbb{T})$ = Space of complex measures on the circle of finite variation $= C(\mathbb{T})^*$

Given $F \in H^1$ $\exists \mu \in M(\mathbb{T})$ s.t. $\hat{\mu}(n) = \int_{\mathbb{T}} \zeta^{-n}\,d\mu(\zeta) = 0 \quad \forall n<0$

and s.t. $F(z) = \int \frac{1-|z|^2}{|1-\bar{z}\zeta|^2}\,d\mu(\zeta)$.

Thm (F. & M. Riesz) If $\mu \in M(\mathbb{T})$ s.t. $\hat{\mu}(n) = 0$ $\forall n<0$ then $d\mu = h\,dt$ (the measure is absolutely continuous) for some h.

Zeroes of H^p functions

Thm If $z_k \in \mathbb{D}$ s.t. $|z_k| \to 1$ as $k\to\infty$ then $\sum(1-|z_k|) < \infty$ iff $\{z_k\}$ is the zero set of an H^p function.

$f \in H^1$ $\exists f^1, f^2 \in H^2$ $\quad f = f^1 f^2$ and $\|f^1\|_2^2 = \|f^2\|_2^2 = \|f\|_1$.

Want to get from the description of zeroes