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<u>Thm.</u> $f \in H^1$, z_k - zeroes of f

$$\Rightarrow \underbrace{\sum (1-|z_k|^2) < \infty}_{\text{"Blaschke condition"}} \quad \underset{\text{equivalent}}{\sim} \quad \sum (1-|z_k|) < \infty$$

since $1-|z_k|^2 = (1-|z_k|)\underbrace{(1+|z_k|)}_{\text{between 1 and 2}}$.

<u>Lemma.</u> Let $f \in Hol(\overline{\mathbb{D}})$ (analytic in bigger disk). $f(z) \neq 0 \ \forall z \in \mathbb{T}$, $f(0) \neq 0$.

z_k : zeroes of f in $\mathbb{D}$.

Then $\sum (1-|z_k|) \underset{\text{Calculus }(**)}{\leq} \sum \log \frac{1}{|z_k|} \underset{(*)}{=} \int_{\mathbb{T}} \log|f(z)| \frac{|dz|}{2\pi} - \log|f(0)|$.

(*): $B(z) = \prod_{k=1}^{n} b_{z_k}(z)$ $\qquad b_{z_k}(z) = \frac{-|z_k|}{z_k} \frac{z-z_k}{1-\bar{z}_k z}$

$g(z) = f(z)/B(z)$.

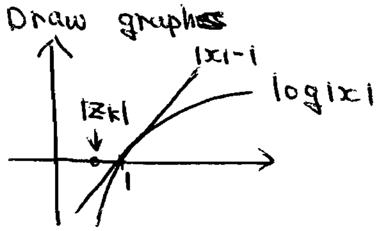
Then, $g(z) \neq 0 \ \forall z \Rightarrow \log g(z) \in Hol(\overline{\mathbb{D}})$

$\Rightarrow \log|g(z)| = Re[\log g(z)] \cdots$ harmonic.

$\underset{\text{Mean Value Property}}{\Rightarrow} \log|g(0)| = \int_{\mathbb{T}} \log|g(z)| \frac{|dz|}{2\pi} \underset{|B(z)|=1 \text{ on } \mathbb{T}}{=} \int_{\mathbb{T}} \log|f(z)| \frac{|dz|}{2\pi}$

$$\log|f(0)| = \sum_{k=1}^{n} \log|b_{z_k}(0)| + \log|g(0)|$$
$$= \sum_{k=1}^{n} \log|z_k| + \int_{\mathbb{T}} \log|f(z)| \frac{|dz|}{2\pi}.$$

(**): Draw graphs



<u>Remark</u> $\int_{\mathbb{T}} \log w(z) \frac{|dz|}{2\pi} \leq \log \int_{\mathbb{T}} w(z) \frac{|dz|}{2\pi}$, $w \geq 0$; Jensen's inequality.

$$\exp \int_{\mathbb{T}} \log w(z) \frac{|dz|}{2\pi} \leq \int_{\mathbb{T}} w(z) \frac{|dz|}{2\pi}$$

"geometric" mean $\leq$ "arithmetic" mean pf of thm;

$f \in H^1(\mathbb{T})$. Apply Lemma to $f(rz)$ $r \in (0,1)$ $f(rz) \neq 0$ $\forall z \in \mathbb{T}$.

Dividing f/z^m, we assume WLOG $f(0) \neq 0$.

$$\sum_{|z_k|<r} \left(1 - \frac{|z_k|}{r}\right) \overset{\text{Lemma}}{\leq} \int_{\mathbb{T}} \log|f(rz)| \frac{|dz|}{2\pi} - \log|f(0)|$$

$$\underset{\text{Jensen's Ineq}}{\leq} \log \underbrace{\int_{\mathbb{T}} |f(rz)| \frac{|dz|}{2\pi}}_{<\infty \text{ because } f \in H^1} - \log|f(0)| \qquad \boxplus$$

$H^p \subset H^1$

So, $f \in H^p \Rightarrow \sum(1-|z_k|) < \infty$.

Blaschke product

$$B(z) = \prod_{k=1}^{\infty} b_{z_k}(z), \quad b_{z_k}(z) = \frac{-|z_k|}{z_k} \frac{z - z_k}{1 - \bar{z}_k z}.$$

Lemma. If $\sum(1-|z_k|) < \infty$, then B converges uniformly on compact subsets.

pf.
$$|1 - b_{z_k}(z)| = \frac{|z_k + z|z_k||}{|z_k||1 - \bar{z}_k z|}(1-|z_k|)$$
$$\leq \frac{1+|z|}{1-|z|}(1-|z_k|)$$
$$\leq \frac{1+r}{1-r}(1-|z_k|) \qquad |z| \leq r < 1.$$

So, $\sum(1-|z_k|) < \infty \Rightarrow \sum|1 - b_{z_k}(z)|$ converges uniformly on compact subsets. $\boxplus$

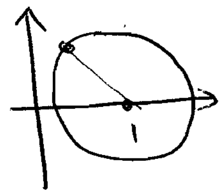
$b_{z_k}(z) \to 1$ uniformly on compact subsets of $\mathbb{D}$.
$\begin{pmatrix} k \to \infty \\ |z_k| \to 1 \end{pmatrix}$

$\log|1+w| \leq C_r|w|$, $|w| \leq r$.

Apply $w = 1 - b_{z_k}(z)$ to get "$\sum \log b_{z_k}$ converges".

$$\log(z) = \int_\gamma \frac{1}{\xi}\, d\xi$$

$$\log(1+w) = \int_\gamma \frac{d\xi}{\xi}$$



$$\Rightarrow \ |\log(1+w)| \leq |w| \frac{1}{1-r}.$$