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Blaschke Condition: $\sum (1-|z_k|) < \infty$.

Lemma: $f \in H^p$, $\{z_k\}$ zeroes of f.

$B = \prod_k b_{z_k}$.

Then $f/B \in H^p$ and $\|f/B\|_{H^p} = \|f\|_{H^p}$.

Pf: Let $B_n = \prod_{k=1}^{n} b_{z_k}$, $g_n = f/B_n$ and $g = f/B$.

$|B_n(z)| \to 1$ as $|z| \to 1$. (since this is true on each factor).

Rewrite this as:

$|B_n(rz)| \to 1$ as $r \to 1^-$ uniformly for $z \in \mathbb{T}$.

Since $\|f\|_{H^p} = \lim_{r \to 1^-} \|f_r\|_{L^p(\mathbb{T})}$; $f_r(z) = f(rz)$.

$\Rightarrow \|f/B_n\|_{H^p} = \|f\|_{H^p}$.

$B_n \to B$ uniformly on compact subsets

$\Rightarrow g_n = f/B_n \to g = f/B$ uniformly on compact subsets.

So for a fixed r, $r \in (0,1)$,

$\|(g_n)_r\|_{L^p} \longrightarrow \|g_r\|_{L^p}$ as $n \to \infty$.

We know $\|(g_n)_r\|_{L^p} \leq \|g_n\|_{H^p} = \|f\|_{H^p}$.

So $\|g_r\|_{L^p} \leq \|f\|_{H^p}$

$\Rightarrow \|g\|_{H^p} \leq \|f\|_{H^p}$.

On the other hand, $|g(z)| \geq |f(z)|$ $\forall z \in \mathbb{D}$, so $\|g\|_{H^p} \geq \|f\|_{H^p}$.

$\therefore \|g\|_{H^p} = \|f\|_{H^p}$.

Corollary: $f \in H^p$, B a Blaschke product.

Then $fB \in H^p$ and $\|fB\|_{H^p} = \|f\|_{H^p}$.

Corollary (Boundary values of Blaschke products).

$B \in H^\infty \subset H^p$. We know that $B_r \to \varphi$ on $\mathbb{T}$ in $L^p(\mathbb{T})$ $1 < p < \infty$.

Claim: $|\varphi| = 1$ almost every where.

Pf of corollary:

Apply Lemma to $f = B$, $g = f/B = 1$.

Then $\|B\|_{H^2} = \|1\|_{H^2} = 1$.

$\|B\|_{H^2} = \|\varphi\|_{L^2}$

$|B(rz)| \le 1 \Rightarrow |\varphi| \le 1$ almost everywhere.

$\Rightarrow$ $|\varphi| = 1$ almost everywhere on $\mathbb{T}$.

Corollary: (Factorization of H^1 functions).

Let $f \in H^1$. Then $\exists\, f_1, f_2 \in H^2$ s.t. $f = f_1 f_2$ and $\|f\|_{H^1} = \|f_1\|_{H^2}^2 = \|f_2\|_{H^2}^2$.

Pf: Let $\{z_k\}$ = zeroes of f. Set $B = \prod b_{z_k}$ and $g = f/B$.

Then $g(z) \neq 0 \ \forall z \in \mathbb{D}$.

Define $f_1 = g^{1/2}$, $f_2 = B g^{1/2}$.

$$\|g^{1/2}\|_{H^2}^2 = \|g\|_{H^1} = \|f\|_{H^1}.$$

$$\|f_2\|_{H^2}^2 = \|g^{1/2}\|_{H^2}^2 = \|f\|_{H^1}.$$

Theorem (Riesz brothers' thm).

$\mu \in M(\mathbb{T})$, F its Poisson extension.

$\hat{\mu}(n) = 0 \ \forall n < 0$.

Then μ is absolutely continuous.

Pf: $\int_{\mathbb{T}} |F(rz)| \frac{|dz|}{2\pi} \le \operatorname{var}\mu \Rightarrow F \in H^1$.

$\rightarrow F = F_1 F_2$, $F_1, F_2 \in H^2$ and $\|F_i\|_{H^2}^2 = \|F\|_{H^1}$ $i = 1, 2$.

$(F_1)_r \to f_1 \in L^2(\mathbb{T})$ in L^2 and $(F_2)_r \to f_2 \in L^2(\mathbb{T})$.

Claim: $(F_1)_r (F_2)_r \to f_1 f_2$ in L^1.

$$\|F_r^1 F_r^2 - F_r^1 f_2 + F_r^1 f_2 - f_1 f_2\|_{L^1} \le \|F_r^1 F_r^2 - F_r^1 f_2\|_{L^1} + \|f_2 (F_r^1 - f_1)\|_{L^1}$$

$$\le \underbrace{\|F_r^1\|_{L^2}}_{\le \|F^1\|_{H^2} \text{ (constant)}} \underbrace{\|F_r^2 - f_2\|_{L^2}}_{\to 0 \text{ as } r \to 1} + \|f_2\|_{L^2} \underbrace{\|F_r^1 - f_1\|_{L^2}}_{\to 0}$$

(Cauchy-Schwartz)

So $\to 0$ as $r \to 1^-$.

A note on the Poisson extension:

$$F(rz) = \int \frac{1-|r|^2}{|1-\bar{z}\xi|^2}\, d\mu(\xi)$$

So $$\int |F(rz)| \frac{|dz|}{2\pi} = \int \left| \int \frac{1-|r|^2}{|1-\bar{z}\xi|^2}\, d\mu(\xi) \right| \frac{|dz|}{2\pi}$$

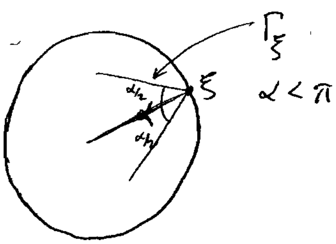
$$\leq \iint \frac{1-r^2}{|1-\bar{z}\xi|^2}\, d|\mu(\xi)|\, \frac{|dz|}{2\pi}.$$

Non-tangentional Convergence to Boundary Values

$\mu \in M(\mathbb{T})$, $\quad d\mu = f\, dt + d\mu_s$.

$\xi \in \mathbb{T}$.

Then $\displaystyle\lim_{\substack{|I|\to 0\\ \xi\in I}} \frac{\mu(I)}{|I|} = f(\xi)$ almost everywhere.



F = Poisson extension of μ.

If $d\mu = f\,\frac{|dz|}{2\pi} + d\mu_s$, $\displaystyle\lim_{\substack{z\to\xi\\ z\in\Gamma_\xi}} F(z) = f(\xi)$ almost everywhere on $\mathbb{T}$.

Non tangential limit (can't approach along a tangent).