2/4/2009 COMPLEX ANALYSIS II

Inner-outer factorization of H^p functions

Let $f \in H^p \subset H^1$

Let $\{z_k\} = \{\text{zeroes of } f\}$, $B = \prod b_{z_k}$ Blaschke product.

Let $g = \frac{f}{B}$. Then last time we proved $\|f\|_p = \|g\|_p$ ($\|\cdot\|_p$ = L^p or H^p norm based on context)

Since g is zero-free, we can take logarithms.

$\mathrm{Re}[\log g(z)] = \ln|g(z)|$ is a harmonic function

Since $\ln(x) \le x$ on $x \in (0,\infty)$, $\ln|g(z)| \le |g(z)|$

Thus $\int_{\mathbb{T}} \ln^+ |g(rz)| \frac{|dz|}{2\pi}$ aside: $\ln^+(x) = \max\{0, \ln(x)\}$ for $x \in (0,\infty)$.

$\le \int_{\mathbb{T}} |\ln|g(rz)|| \frac{|dz|}{2\pi} \le \|f\|_1$

Also $\int_{\mathbb{T}} \ln|g(rz)| \frac{|dz|}{2\pi} = \ln|g(0)|$ by mean-value theorem

So $\int \ln^- |g(rz)| \frac{|dz|}{2\pi}$ will also be uniformly bounded.

Thus $\int_{\mathbb{T}} \left| \ln|g(rz)| \right| \frac{|dz|}{2\pi} \le C$

Thus $(\ln|g(r_k z)|) \xrightarrow[\text{weak-*}]{} \nu$ in $M(\mathbb{T})$. By Banach-Alaoglu.

signed Borel measures of finite variation (real valued since sequence is real valued)

$r_k \to 1$

And, $\ln|g(z)|$ = Poisson integral = $\int \frac{1-|z|^2}{|1-\bar{z}\xi|^2} d\nu(\xi)$

How do we find the harmonic conjugate?

Note $\frac{\xi + z}{\xi - z}$ has real part $\frac{1-|z|^2}{|1-\bar{\xi}z|^2}$ = Poisson kernel.

complex conjugates

How did we get this? Recall $\frac{1-|z|^2}{|1-\bar{\xi}z|^2} = 1 + \sum_{k>0} z^k \bar{\xi}^k + \sum_{k>0} \bar{z}^k \xi^k$

So we take $1 + 2\sum_{k>0} z^k \bar{\xi}^k$. Real part of this is Poisson kernel

This $= 1 + 2\dfrac{z\bar{\xi}}{1-z\bar{\xi}} = 1 + 2\dfrac{z}{\xi - z} = \dfrac{\xi - z}{\xi - z} + \dfrac{2z}{\xi - z} = \dfrac{\xi + z}{\xi - z}$

So $\log g(z) = \displaystyle\int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\nu(\xi) + iR$ where R is a real constant.

By Lebesgue decomposition, we can write

$$d\nu = w(\xi)\frac{|d\xi|}{2\pi} + \underbrace{d\nu_s}_{\text{singular part}}$$

$$= \ln|g(\xi)|\frac{|d\xi|}{2\pi} + d\nu_s$$ This follows from our discussion of non-tangential limits.

Thus $w(\xi) \underset{a.e}{=} \lim_{\substack{z\to\xi \\ \text{non-tangential}}} \ln|g(z)| = \ln|g(\xi)| = \ln|f(\xi)|$

Claim: $\nu_s \le 0$

Pf: $\ln|g(z)| \le |g(z)| \Rightarrow (\ln|g|)_r \le |g|_r \longrightarrow |g|\frac{|d\xi|}{2\pi}$ since $g_r \to g$ in L^1.

$(\ln|g|)_r \downarrow \nu$

Thus $\nu \le |g|\frac{|d\xi|}{2\pi}$. Thus it cannot have positive singular part.

You cannot dominate a positive singular measure by an absolutely continuous one. □

Now we're ready to write down inner-outer factorization

i.e. if $f \in H^p$, then $f = \alpha B S F$ where α = unimodular constant

B = Blaschke product

$S = \exp \displaystyle\int_{\mathbb{T}} \frac{z+\xi}{z-\xi}\, d\mu(\xi)$ where $\mu \ge 0$ singular

in our case $\mu = -\nu_s$

$F = \exp \displaystyle\int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \ln|f(\xi)| \frac{|d\xi|}{2\pi}$

F = outer part of f

BS = inner part of f

S = singular inner part of f

Remark: $|B| = |S| = 1$ a.e. on $\mathbb{T}$. We already showed this for B.

$$|S(\xi)| = \lim_{\substack{z\to\xi \\ \text{non-tang}}} \exp\left\{-\int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\xi|^2}\, d\mu(\xi)\right\} = \exp\left\{\underbrace{\lim_{\substack{z\to\xi \\ \text{non-tang}}} -\int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\xi|^2}\, d\mu(\xi)}_{=0}\right\}$$

because we're taking Poisson extension of a singular measure

<u>Remark:</u> $\frac{1}{S} \notin H^p \ \forall p$, but $\left|\frac{1}{S}\right| = 1$ a.e. on $\mathbb{T}$

Hankel operators

Hankel matrices:
$$\begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_2 & \gamma_3 & \gamma_4 & \ddots \\ \gamma_3 & \gamma_4 & \ddots & \\ \gamma_4 & & & \end{bmatrix}$$
infinite matrices that are constant on top-right $\to$ bottom-left diagonals.

How does this relate to Hardy spaces.

Consider H^2

$$H^2_- = L^2 \ominus H^2 = \text{ortho complement of } H^2 \text{ in } L^2$$

Let φ be a function on $\mathbb{T}$

$P_+ = P_{H^2}$, $P_- = P_{H^2_-}$ orthogonal projections

Define $H_\varphi : H^2 \longrightarrow H^2_-$

$f \longmapsto P_-(\varphi f)$

P_- " negative fourier coeffs.

We have a standard basis in H^2 of $\{z^n\}_{n\geq 0}$

" " in H^2_- of $\{z^{-n}\}_{n<0} = \{\bar{z}^n\}_{n>0}$ since we're on $\mathbb{T}$

<u>Claim:</u> H_φ has matrix
$$\begin{bmatrix} \hat\varphi(-1) & \hat\varphi(-2) & \hat\varphi(-3) \\ \hat\varphi(-2) & \hat\varphi(-3) & \\ \hat\varphi(-3) & & \ddots \end{bmatrix}$$
This is clear.

<u>Remarks:</u> If $\varphi \in L^\infty(\mathbb{T})$, then H_φ is a bounded operator, and $\|H_\varphi\| \leq \|\varphi\|_\infty$

Since $\|M_\varphi\| = \|\varphi\|_\infty$. $H_\varphi = P_- M_\varphi$, $\|H_\varphi\| \leq \|P_-\|\,\|M_\varphi\| = \|M_\varphi\|$ since $\|P_-\| = 1$

In fact we have equality $\|H_\varphi\| = \|\varphi\|_\infty$ for some φ.