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Diana Davis

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Theorem (Nehari Theorem)

Let $\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_2 & \\ \gamma_3 & & \ddots \end{pmatrix}$ be bounded. Then there exists $\varphi \in L^\infty(\mathbb{T})$ such that
$\hat{\varphi}(-k) = \gamma_k$ for all $k > 0$.

bounded operator in ℓ^2

$(\Gamma = H_\varphi)$ and such that $\|\varphi\|_\infty = \|H_\varphi\|)$

Proof

Define $\varphi_- = \sum_{k>0} \gamma_k \bar{z}^k$, $z \in \mathbb{T}$.

Γ is bounded $\Rightarrow \sum |\gamma_k|^2 < \infty \Rightarrow \varphi_- \in L^2(\mathbb{T})$.

Define an operator $H_{\varphi_-} : H^2 \cap H^\infty \longrightarrow H^2_-$ whose matrix is Γ

Define a bounded linear functional

$$L : zH^1 \cap H^\infty \longrightarrow \mathbb{C}, \quad Lzf = \int_{\mathbb{T}} \varphi_- zf \frac{|dz|}{2\pi}, \quad f \in H^\infty.$$

$f \in H^\infty \subset H^1$

$f = f_1 f_2$, $f_1, f_2 \in H^2$

$\|f_1\|_2^2 = \|f_2\|_2^2 = \|f\|_1 = \|zf\|_1$

$L(zf) = \int \varphi_- f_1 \overline{\overline{z f_2}} \frac{|dz|}{2\pi}$ double complex conjugate so that we can write it as an inner product:

$$= (\varphi_- f_1, \underbrace{\overline{z f_2}}_{H^2_-}) = (P_-(\varphi_- f_1), \overline{z f_2}) = (H_{\varphi_-} f_1, \overline{z f_2})$$

by definition of the Haenkel operator

Haenkel operator is bounded, so

$$|L(zf)| \leq \|\Gamma\| \, \|f_1\|_2 \, \|z f_2\|_2 = \|\Gamma\| \, \|f\|_1$$

L is a bounded function, so L can be extended by Hahn-Banach to a bounded linear functional on L^1.
to $\tilde{L} \in (L^1)^*$

So there exists $\varphi \in L^\infty$, $\|\varphi\| = \|L\| \leq \|\Gamma\|$ such that

$$\int \varphi z f \frac{|dz|}{2\pi} = \int \varphi_- z f \frac{|dz|}{2\pi} \quad \text{for all } f \in H^1.$$

$$\hat{\varphi}(-n) = \int \varphi z^n \frac{|dz|}{2\pi} = \int \varphi_- z^n \frac{|dz|}{2\pi} \quad \text{for all } n > 0.$$

$\hat{\varphi}(-n) = \hat{\varphi}_-(-n)$ for all $n > 0$

So $H\varphi = H\varphi_-$,

$\|\varphi\|_\infty \le \|\Gamma\| = \|H\varphi\|$

and $\geq$ is trivial so we have equality.

Informally, $\text{dist}_{L^\infty}(\varphi, H^\infty) = \|H\varphi\|$.

Application

Given $a_0, \ldots, a_n$ find $\min\{\|f\|_\infty : f \in H^\infty : f^{(k)}(0) = a_k,\ 0 \le k \le n\}$.

"Given the first $n+1$ terms of the Taylor expansion, what is the minimum norm possible?"

HW Find finite Hankel matrix whose norm gives the desired minimum.

Hint: $f_0 = \sum_{k=0}^{n} \frac{a_k}{k!} z^k$, want to find $\text{dist}(f, z^n H^\infty)$.

If you reduce this problem to H^∞ you will find the desired matrix, consisting of mostly zeroes except in a finite block.

Thm $H\varphi$ such that $\|\varphi\|_\infty = \|H\varphi\|$

Let there exist $f \in H^2$ such that $\|H\varphi f\|_2 = \|H\varphi\| \cdot \|f\|_2$.

Then $\varphi = \frac{H\varphi f}{f}$. So if the norm of $H\varphi$ is attained, the best φ is unique.

Proof: Exercise.

Reproducing Kernels

Let $\mathcal{H}$ be a Hilbert space of analytic functions in some domain Ω.

For all λ, $\lambda : f \longmapsto f(\lambda)$
$f \in \mathcal{H}$ is bounded.

So there exists a vector $K_\lambda \in \mathcal{H}$ such that $f(\lambda) = (f, K_\lambda)_{\mathcal{H}}$.

K_λ is a function, $K_\lambda(z) = K(z, \lambda)$.

Example $\mathcal{H} = H^2$

$$K_\lambda(z) = \frac{1}{1 - \bar{\lambda} z}$$

Proof $f \in \mathrm{Hol}(\bar{\mathbb{D}})$

$$(f, K_\lambda) = \int_{\mathbb{T}} f(z) \frac{1}{1 - \lambda \bar{z}} \frac{dz}{2\pi i z} \quad \text{because } |dz| = \frac{dz}{iz}$$

$$= \frac{1}{2\pi i} \int f(z) \frac{1}{z - \lambda} \, dz = f(\lambda) \quad \text{Cauchy formula}$$

Finding K_λ in the general case

Assume $\{e_n\}_{n=0}^\infty$ is an orthonormal basis in $\mathcal{H}$.

Then $f \in \mathcal{H} \Rightarrow f = \sum_{n=0}^\infty (f, e_n) e_n$ standard formula for Fourier series

$$f(\lambda) = \sum_{n=0}^\infty (f, e_n) \, e_n(\lambda)$$

If $f \longmapsto f(\lambda)$ is bounded, then $\sum |e_n(\lambda)|^2 < \infty$ (a standard exercise).

Then $f(\lambda) = \left(f, \sum \overline{e_n(\lambda)} \, e_n\right)$ the bar appears because $(f, \alpha g) = \bar{\alpha}(f, g)$.

$$K_\lambda = \sum \overline{e_n(\lambda)} \, e_n$$

$$K_\lambda(z) = K(z, \lambda) = \sum e_n(z) \overline{e_n(\lambda)}$$

If $\mathcal{H} = H^2$ then $e_n(z) = z^n$, $n \geq 0$ standard orthonormal basis

$$K(z, \lambda) = \sum z^n \bar{\lambda}^n = \frac{1}{1 - \lambda z}$$

Example Bergman space

$$A^2 = \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \underbrace{\iint_{\mathbb{D}} |f(z)|^2 \, dx \, dy}_{\|f\|^2_{A^2}} < \infty \right\}$$

Exercise Compute the reproducing kernel for the Bergman space A^2.

Hint: $\{z^n\}_{n \geq 0}$ is not an orthonormal basis, but it is an orthogonal basis.

Remark $\|K_\lambda\| = (K_\lambda, K_\lambda) = K_\lambda(\lambda) = K(\lambda, \lambda)$.