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Last time introduced reproducing kernels in Hilbert space

$\mathcal{H}$ $\quad K_\lambda : (f, K_\lambda) = f(\lambda)$

$K_\lambda(z) = K(z, \lambda)$

If $\mathcal{H} = H^2$, $\quad K(z, \lambda) = \frac{1}{1 - \bar{\lambda} z}$.

Properties of repr. kernels

1. $K(z, w)$ analytic in $z, \bar{w}$ $\quad$ (anti-analytic in w) $\left\{ \begin{array}{l} \text{(holomorphic in } z \text{ since } K_\lambda \in \mathcal{H}, \\ (\text{" " } \bar{w} \text{ because of 2.)} \end{array} \right.$
2. $K(z, w) = \overline{K(w, z)}$.

This follows from the formula $K(z, w) = \sum e_k(z) \overline{e_k(w)}$.

3. K p.s.d.. That is, $\forall n$, $\forall z_1, \dots, z_n \in \mathbb{D}$ the matrix

$$\{K(z_k, z_j)\}_{\substack{k=1 \\ j=1}}^{\infty} \geq 0 \qquad \text{(p.s.d.} \equiv \text{positive semi-definite)}$$

i.e. $\forall \alpha_1, \dots, \alpha_n \in \mathbb{C}$, $\displaystyle\sum_{k=1}^{n} \sum_{j=1}^{n} K(z_k, z_j) \alpha_j \bar{\alpha}_k \geq 0$.

Pf $\quad K(z_k, z_j) = (K_{z_j}, K_{z_k})_{\mathcal{H}}$
$\qquad \| $
$\quad K_{z_j}(z_k)$

$$0 \leq \left\| \sum \alpha_j K_{z_j} \right\|^2 = \left(\sum_j \alpha_j K_{z_j}, \sum_k \alpha_k K_{z_k} \right)$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \bar{\alpha}_k (K_{z_j}, K_{z_k})$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \alpha_j \bar{\alpha}_k K(z_k, z_j).$$

Rk: $\{K(z_k, z_j)\}_{k,j=1}^{n}$ is called the Gram matrix.
It is a matrix of inner product, (of form (f_j, f_k))
and is always p.s.d.

Given $K(z,w)$ satisfying properties 1–3 we can construct Hilbert space of analytic functions s.t. K is its repr. kernel.

Define $K_\lambda(z) = K(z,\lambda)$.

For $z_1, \dots, z_n$, $\alpha_1, \dots, \alpha_n$ define $\left\| \sum_{k=1}^{n} \alpha_k K_{z_k} \right\|^2 = \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_j \bar{\alpha}_k K(z_k, z_j)$,

Take quotient space (if necessary) and complete.

<u>Uniqueness theorem for H^p</u>: If $f \in H^p$ then $\int_{\mathbb{T}} \ln |f| \frac{|dz|}{2\pi}$ converges.

<u>Remark</u> $\ln x \le x$ for $x > 0$, so $p \ln x \le x^p$,

and so $\int \ln |f| \le \frac{1}{p} \int |f|^p$, and the same holds for $\ln^+$.

If $f \in L^p(\mathbb{T})$ then $\int \ln|f|$ diverges iff $\int_{\mathbb{T}} \ln |f| = -\infty$.

Restating the theorem: If $f \in H^p$, $\int_{\mathbb{T}} \ln |f| = -\infty$

then $|f| \equiv 0$ a.e. $f \equiv 0$ on $\mathbb{D}$.

In particular, if $f \equiv 0$ on $E \subset \mathbb{T}$, $m(E) > 0 \Rightarrow f \equiv 0$.

In fact we already have the proof from inner-outer factorization.

If $g = f_0$ then $g(0) = \exp \left\{ \int \ln |f| \frac{|dz|}{2\pi} \right\}$

$$\int_{\mathbb{T}} \ln^+ |f| \frac{|dz|}{2\pi} \le \frac{1}{p} \int |f|^p \frac{|dz|}{2\pi}$$

so $$\int \ln^- |f| \frac{|dz|}{2\pi} > -\infty$$

In summary, the boundary values of an analytic fn have traces of analyticity.

How "bad" can the boundary values be?

<u>Thm</u> Take any $F \in L^p$, $F \geq 0$ s.t. $\int_{\mathbb{T}} \ln |F| \frac{|dz|}{2\pi} > -\infty$.

then $\exists f \in H^p$ s.t. $|f| = F$ a.e. on $\mathbb{T}$.

<u>Pf</u> let $f(z) = \exp\left\{ \int \ln F(\zeta) \frac{\zeta + z}{\zeta - z} \frac{|d\zeta|}{2\pi} \right\}$.

Given points $z_1, \dots, z_n \in \mathbb{D}$, $w_1, \dots, w_n \in \mathbb{C}$,

$f \in H^\infty$ s.t. $f(z_k) = w_k$.

"Favorite" two operators: ① <u>Shift operator</u> $S : H^2 \to H^2$, $Sf = zf$.

$$\sum_{k \geq 0} \hat{f}(k) z^k \mapsto \sum_{k \geq 0} \hat{f}(k) z^{k+1}$$

$$(\hat{f}(0), \hat{f}(1), \dots) \mapsto (0, \hat{f}(0), \hat{f}(1), \dots)$$

② <u>Backward shift</u> $S^* : H^2 \to H^2$ $\qquad S^* f = \frac{f - f(0)}{z} = \frac{f - \hat{f}(0)}{z}$

$$\sum_{k \geq 0} \hat{f}(k) z^k \longmapsto \sum_{k \geq 1} \hat{f}(k) z^{k-1}$$

Then $(Sf, g) = (f, S^* g)$ $\forall f \in H^2$.

$S^* K_\lambda = \bar{\lambda} K_\lambda \quad \forall \lambda \in \mathbb{D}$

S has no eigenvectors.

Models of operators, representing operators in terms of operators we know well.

Def $f \in H^p$ is called <u>inner</u> if $|f| = 1$ a.e. on $\mathbb{T}$.

Rk If $f \in H^p$ is inner then $f \in H^\infty$.

Note duality between fns in the disk and on the boundary.

Thm (Beurling): Let $\mathcal{E} \subset H^2$ be a closed (non zero) subspace, $S\mathcal{E} \subset \mathcal{E}$.
Then $\exists$ inner $\theta \in H^\infty$ s.t. $\mathcal{E} = \theta H^2$
(i.e. any fn in $\mathcal{E}$ can be written θf for some $f \in H^2$.)
Also, θ is unique up to mult. constant, (i.e. if $\mathcal{E} = \theta_1 H^2$ for some θ_1 inner then $\theta_1 = \alpha\theta$ for some $\alpha \in \mathbb{T}$).