①

Beurling Theorem

$\{0\} \neq \mathcal{E} \subset H^2$, $z\mathcal{E} \subset \mathcal{E}$. Then $\exists$ inner $\theta \in H^\infty$ s.t. $\mathcal{E} = \theta H^2$, and θ is "unique"

Pf uniqueness: Suppose $\mathcal{E} = \theta H^2 = \theta_1 H^2$, θ, θ_1 both inner ($|\theta(z)| = 1$ for $z \in \mathbb{T}$).

$\Rightarrow \theta_1 \in \mathcal{E} \Rightarrow \theta_1 = \theta f$, $f \in H^2$.

so $\theta_1 \bar{\theta} = f$ on $\mathbb{T}$, $f \in H^2$.

Similarly $\theta \in \mathcal{E} \Rightarrow \theta = \theta_1 g$, $g \in H^2$.

so $\theta \bar{\theta}_1 = g \in H^2$ (on $\mathbb{T}$).

$\theta_1 \bar{\theta} = \sum_{k \geq 0} a_k z^k$, and $\theta \bar{\theta}_1 = \overline{\theta_1 \bar{\theta}} = \sum_{k \geq 0} \bar{a}_k \bar{z}^k \in H^2$

so $\theta_1 \bar{\theta} = \bar{a}_0$.

Existence: 1. $\bigcap_{n \geq 0} S^n \mathcal{E} \subset \bigcap_{n \geq 0} S^n H^2 = \{0\}$.

Therefore, $S\mathcal{E} \subsetneq \mathcal{E}$.

2. $S\mathcal{E} \neq \mathcal{E} \Rightarrow \exists \theta \in \mathcal{E}$, $\theta \perp S\mathcal{E}$, $\theta \neq 0$.

WLOG, assume $\|\theta\|_{H^2} = 1$.

$z\theta \in z\mathcal{E} = S\mathcal{E} \perp \theta$.

So $z\theta \perp \theta$

But also $z^n \theta \perp \theta$ $\forall n \geq 1$. (since $z\mathcal{E} \subset \mathcal{E}$, $z^n \theta \in z^n \mathcal{E} \subset S\mathcal{E}$).

$\therefore \int_{\mathbb{T}} z^n \theta \bar{\theta} \frac{|dz|}{2\pi} = 0$ $\forall n \geq 0$. and $\int_{\mathbb{T}} \bar{z}^n |\theta|^2 \frac{|dz|}{2\pi} = 0$ $\forall n \geq 0$.

So $\int_{\mathbb{T}} z^n |\theta|^2 \frac{|dz|}{2\pi} = 0$ $\forall n \neq 0$, and if $n = 0$, we get $\int_{\mathbb{T}} |\theta|^2 \frac{|dz|}{2\pi} = 1$.

Therefore, $|\theta|^2 = 1$ almost everywhere, and θ is inner,

[Note that M_z = multiplication by z is invertible on $L^2(\mathbb{T})$, so $z\mathcal{E}$ is a closed subset of H^2 $\rightsquigarrow$ $z\mathcal{E}$ isn't dense in $\mathcal{E}$, and we really can find θ in the orthogonal complement]

3. $\theta H^2 = \operatorname{Clos} \theta \mathrm{Pol}_A \subset \mathcal{E}$

$p \in \mathrm{Pol}_A$; means $p = \sum_{k \geq 0} a_k z^k$ finite sum

$p \in \mathrm{Pol}_A \Rightarrow \theta p \in \mathcal{E}$.

4. $\mathcal{E} \subset \theta H^2$.

$f \in \mathcal{E} \Rightarrow z^n f \in z\mathcal{E} \perp \theta$ $\forall n \geq 1$.

$\Rightarrow \int_{\mathbb{T}} \theta \overline{f z^n} \frac{|dz|}{2\pi} = 0$ $\forall n \geq 1$.

Pf of Existence cont.

Assume $f \perp \Theta H^2$. Then $f \perp \Theta z^n \ \forall n \geq 0 \Rightarrow \int_{\mathbb{T}} \theta\, z^n \bar{f} \frac{|dz|}{2\pi} = 0 \ \forall n \geq 0$.

Therefore, $\int_{\mathbb{T}} \theta \bar{f} z^n \frac{|dz|}{2\pi} \quad \forall n \in \mathbb{Z}$.

$\Rightarrow \theta \bar{f} = 0$ a.e. on $\mathbb{T}$ $\Rightarrow f = 0$ a.e. on $\mathbb{T}$ (since $|\theta| = 1$ a.e. on $\mathbb{T}$) $\Rightarrow f = 0$

So there is no f, $f \perp \Theta H^2$, and $\mathcal{E} \subset \Theta H^2$.

<u>Remark</u>: $\dim \mathcal{E} \ominus S\mathcal{E} = 1$.

Follows from uniqueness.

<u>Arithmetic of Inner Functions and Geometry of Inner Functions</u>.

<u>Theorem</u>: $\mathcal{E}_1 = \theta_1 H^2$, $\mathcal{E}_2 = \theta_2 H^2$. Then $\mathcal{E}_2 \subset \mathcal{E}_1$ iff $\theta_2 = \theta_1 \theta$, θ an inner function. ($\theta \in H^\infty$).

Pf: $\theta_2 \in \mathcal{E}_2 \subset \mathcal{E}_1 = \theta_1 H^2 \Rightarrow \exists \theta \in H^2$ s.t. $\theta_2 = \theta_1 \theta$.

$|\theta| = 1$ on $\mathbb{T}$ almost everywhere.

$\theta \in H^\infty$ since $\theta \in H^2$ and boundary values in L^∞.

$\theta_2 H^2 = \theta_1 \theta H^2 \subset \theta_1 H^2 = \mathcal{E}_1$.

<u>Corollary</u>: $f \in H^2$ $f = B \cdot S \cdot f_0 = f_i \cdot f_o$ (B - Blaschke product, f_o - outer part).

$\mathcal{E}_f := \text{span}\{z^n f : n \geq 0\}$. (closed span).

Then $\mathcal{E}_f = f_i H^2$.

Pf: $\mathcal{E}_f = \Theta H^2$ for some Θ inner.

If $p \in Pol_A$, then $fp = f_i f_o p \in f_i H^2$.

$\{fp : p \in Pol_A\}$ is dense in $\mathcal{E}_f = \Theta H^2$.

$\Rightarrow \Theta H^2 = \mathcal{E}_f \subset f_i H^2$. since $f_i H^2$ is closed.

$\Rightarrow \Theta = f_i \theta_1$ for some $\theta_1 \in H^2$.

Also, $f \in \mathcal{E}_f$, so $f = \Theta h$, some $h \in H^2$.

$f_i f_o = f_i \theta_1 h \Rightarrow f_o = \theta_1 h$. Which is impossible.

(look at inner-outer factorization — f_o is outer, but θ_1 has nontrivial inner part).

so $\theta_1 = 1$, and $\mathcal{E}_f = f_i H^2$.

<u>Corollary</u>: $f \in H^2$ is outer iff $\mathcal{E}_f = H^2$.