2/13/2009 COMPLEX ANALYSIS

Model operators + Toeplitz operators

Θ inner, so ΘH^2 is forward-shift invariant

Let $K_\Theta = H^2 \ominus \Theta H^2$ (ortho complement)

Then $S^* K_\Theta \subset K_\Theta$ i.e. it is backward shift invariant.

From functional analysis, it follows that $AE \subset E \iff A^* E^\perp \subset E^\perp$ for E closed

Define $T = S^*|_{K_\Theta}$ This is our model operator.

Recall: Toeplitz operators

$\varphi \in L^\infty(\mathbb{T})$ $T_\varphi : H^2 \to H^2$ $T_\varphi f = P_+(\varphi f)$ projection onto H^2 from L^2

1. $\|T_\varphi\| = \|\varphi\|_\infty$

   $\leq$ is trivial since $\|T_\varphi f\|_2 = \|P_+(\varphi f)\|_2 \leq \|\varphi f\|_2 \leq \|\varphi\|_\infty \|f\|_2$

2. Matrix of T_φ

$$\begin{pmatrix} \hat{\varphi}(0) & \hat{\varphi}(-1) & \hat{\varphi}(-2) \\ \hat{\varphi}(1) & \hat{\varphi}(0) & \hat{\varphi}(-1) & \cdots \\ \hat{\varphi}(2) & \hat{\varphi}(1) & \hat{\varphi}(0) \end{pmatrix}$$

Unlike Hankel operators, φ determines T_φ for Toeplitz operators.

3. $T_\varphi^* = T_{\bar{\varphi}}$ because $(T_\varphi f, g) = (P_+ \varphi f, g) = (\varphi f, P_+ g) = (\varphi f, g)$

$$\stackrel{?}{=} (f, \bar{\varphi} g) = (P_+ f, \bar{\varphi} g) = (f, P_+ \bar{\varphi} g) = (f, T_{\bar{\varphi}} g)$$

Why the ? Because φ isn't technically an operator on H^2

But $(\varphi f, g) = \int \varphi f \bar{g} = \int f \overline{\bar{\varphi} g} = (f, \bar{\varphi} g)$

Then we see $S = T_z$, $S^* = T_{\bar{z}}$

4. Let $\varphi, \psi \in H^\infty$.

Then (a) $T_\varphi T_\psi = T_{\varphi\psi} = T_{\psi\varphi} = T_\psi T_\varphi$

(b.) $T_{\bar{\psi}} T_{\bar{\varphi}} = T_{\bar{\psi}\bar{\varphi}} = T_{\bar{\varphi}\bar{\psi}} = T_{\bar{\varphi}} T_{\bar{\psi}}$

This follows from: For part b, take adjoints.

5. For $\varphi \in H^\infty$, $T_\varphi f = \varphi f$.

Lemma: If $h \in H^\infty$, $K_\lambda = \frac{1}{1-\bar\lambda z}$ is the reproducing kernel for H^2, then $T_{\bar h} K_\lambda = \overline{h(\lambda)} K_\lambda$.

Pf: Exercise.

Hint: $\bar h K_\lambda - \overline{h(\lambda)} K_\lambda \perp H^2$

Commutant lifting theorem

Suppose Θ inner, $T = S^*|_{K_\Theta}$ wher $K_\Theta = H^2 \ominus \Theta H^2$

We want to describe all $A: K_\Theta \to K_\Theta$ that commute with T, i.e. $AT = TA$

Example of such: $T_{\bar h}|_{K_\Theta}$ where $h \in H^\infty$. It's an operator on K_Θ since $T_h \Theta H^2 \subset \Theta H^2$
$\Rightarrow T_{\bar h} K_\Theta \subset K_\Theta$.

It commutes w/ S^* since $S^* = T_{\bar z}$.

The commutant lifting theorem says these are all examples.

Theorem (D. Sarason) (More general version: B. Sz.Nagy - C. Foias)

If $T = S^*|_{K_\Theta}$ and $A: K_\Theta \to K_\Theta$ s.t. $TA = AT$, then $\exists \bar h \in H^\infty$ s.t. $A = T_{\bar h}|_{K_\Theta}$ and $\|h\|_\infty = \|A\|$.

Corollary (Pick's Theorem)

Let $\{z_i\}_1^n$, $\{w_i\}_1^n$ be points in $\mathbb{D}$

Then TFAE:

1.) $\exists h \in H^\infty$ with $\|h\| \le 1$ s.t. $h(z_k) = w_k$

2.) The matrix $\left(\frac{1-\bar w_i w_j}{1-\bar z_i z_j}\right)$ is positive semidefinite.

Pf: Consider $\mathcal{K} = \text{span}\{K_{z_i} \mid i = 1, \dots, n\}$. Then $S^*\mathcal{K} \subset \mathcal{K}$. Thus $\mathcal{K} = K_\Theta$ for some Θ inner.

To prove $1 \Rightarrow 2$, introduce $A: \mathcal{K} \to \mathcal{K}$ defined by $Af = P_{\mathcal{K}} \bar h f$ where h is given by (1)

Then $AK_{z_i} = T_{\bar h} K_{z_i} = \overline{h(z_i)} K_{z_i} = \bar w_i K_{z_i}$

We know $\|A\| \le \|T_{\bar h}\| = \|h\|_\infty \le 1$. So A is a contraction.

$\Updownarrow$

$\forall \alpha_1, \dots \alpha_n \in \mathbb{C}$, we have $\|A \sum \bar\alpha_i K_{z_i}\|^2 \le \|\sum \bar\alpha_i K_{z_i}\|^2$

$\Updownarrow$

$$\Big\|\sum \bar\alpha_i \bar w_i K_{z_i}\Big\|^2 \le \Big\|\sum \bar\alpha_i K_{z_i}\Big\|^2 = \sum_{i,j} \bar\alpha_i \alpha_j (K_{z_i}, K_{z_j}) = \sum_{i,j} \bar\alpha_i \alpha_j \frac{1}{1-\bar z_i z_j}$$

$\parallel$

$$\sum \frac{\bar\alpha_i \bar w_i \alpha_j w_j}{1-\bar z_i z_j} \iff \sum \bar\alpha_i \alpha_j \frac{1-\bar w_i w_j}{1-\bar z_i w_j} \ge 0.$$ So $1 \Rightarrow 2$. $\square$

Conversely, to prove $2 \Rightarrow 1$.

Define: $A : \mathcal{K} \to \mathcal{K}$ by $A K_{z_i} = \bar{w}_i K_{z_i}$

By the proof $1 \Rightarrow 2$, positive semidefinite-ness of the matrix in (2) $\Leftrightarrow$ $\|A\| \le 1$.

By the commutant lifting theorem, $\exists h \in H^\infty$ s.t. $A = T_{\bar{h}}|_{\mathcal{K}}$. $\|h\|_\infty = \|A\| \le 1$

$$\text{Then } \begin{array}{l} A K_{z_k} = \bar{w}_k K_{z_k} \\ \quad \| \\ T_{\bar{h}} K_{z_k} = \overline{h(z_k)} K_{z_k} \end{array} \quad \Rightarrow h(z_k) = w_k. \square$$