Commutant Lifting Theorem

$T = S^*|_{\mathcal{K}}$, where $\mathcal{K} = K_\theta$, θ inner.

A an operator s.t. $AT = TA$. Then $AT^2 = TAT = T^2A$, and more generally $T^nA = AT^n$

$T^n = S^*|_{\mathcal{K}}$

Define $T^* = P_{\mathcal{K}} S|_{\mathcal{K}}$. $T^{*n} = P_{\mathcal{K}} S^n|_{\mathcal{K}}$; and $T^{*n}A^* = A^*T^{*n}$ $\forall n \geq 1$

Lemma: Let $f \in H^2$. Then 1. $f \in K_\theta \Leftrightarrow \bar{\theta} f \in H^2_-$

2. $P_{K_\theta} f = \theta P_-(\bar{\theta} f)$.

Pf: 1. $f \in K_\theta \Leftrightarrow (f, \theta g) = 0 \quad \forall g \in H^2$

$\Leftrightarrow (\bar{\theta} f, g) = 0 \quad \forall g \in H^2$

$\Leftrightarrow \bar{\theta} f \perp H^2 \Leftrightarrow \bar{\theta} f \in H^2_-$.

2. $f = f_1 + f_2$, $f_1 \in P_{K_\theta}$, $f_2 \in \theta H^2$.

Then $\theta P_-(\bar{\theta} f_1) = \theta \bar{\theta} f_1 = f_1$.

and $\theta P_-(\bar{\theta} f_2) = 0$

Definition: $\Gamma : H^2 \to H^2_-$

$\Gamma f = 0$ if $f \in \theta H^2$.

$\Gamma f = \bar{\theta} A^* f$, $f \in K_\theta$.

Claim: Γ is a Hankel operator.

Pf: Let $\varphi_0 = \Gamma \cdot 1 = \bar{\theta} A^* P_{K_\theta} 1 \in H^2_-$.

$$\Gamma z^n = \Gamma S^n 1 = \bar{\theta} A^* P_{K_\theta} S^n 1 = \bar{\theta} A^* P_{K_\theta} S^n P_{K_\theta} 1.$$

$$= \bar{\theta} A^* T^{*n} P_{K_\theta} 1 = \bar{\theta} P_{\mathcal{K}} S^n A^* P_{K_\theta} 1$$

$$= \bar{\theta} \theta P_-(z^n \bar{\theta} A^* P_{K_\theta} 1)$$

$$= P_-(z^n \bar{\theta} A^* P_{K_\theta} 1)$$

$$= P_-(z^n \varphi_0).$$

For any polynomial p, $\Gamma p = P_-(\varphi_0 p)$.

By Nehari Theorem, $\exists \varphi \in L^\infty$ s.t. $\|\varphi\|_\infty = \|\Gamma\|$ and $\Gamma = H_\varphi$.

Set $h = \varphi \cdot \theta$, so φ is $\bar{\theta} h$. $\|h\|_\infty = \|\varphi\|_\infty$

If $f \in H^2$, $\Gamma\theta f = 0$, and also $\Gamma\theta f = H_{\bar\theta h}\theta f = P_-(hf)$

$\Rightarrow P_-(fh) = 0 \ \forall f \in H^2$

setting $f = z^n, n>0$, we see that $\hat{h}(n) = 0 \ \forall n>0 \Rightarrow h \in H^\infty$.

$f \in K_\theta$: Then $A^* f = \theta \Gamma f$

$$= \theta P_- \bar\theta h f = P_{K_\theta} h f.$$

$$\Rightarrow A f = P_{K_\theta} \bar h f = P_+ \bar h f.$$

$$\overset{\shortparallel}{P_{K_\theta} \underbrace{P_+ \bar h f}_{K_\theta}}$$

$$\left(\begin{array}{l} A^* = P_{K_\theta} T_h |_{K_\theta} \Rightarrow A = P_{K_\theta} T_{\bar h} |_{K_\theta} \\ T_h \theta H^2 \subset \theta H^2 \\ T_{\bar h} K_\theta \subset K_\theta \end{array} \right)$$

<u>Question</u>: When (under what conditions for z_k) does the interpolation problem
$f(z_k) = w_k \quad k = 1, 2, \cdots$ have a solution $\forall \{w_k\} \in \ell^\infty$?

<u>Definition</u>: sequences $\{z_k\}_{k=1}^\infty$ s.t. the problem:

$$(*) \quad \exists f \in H^\infty \text{ s.t. } f(z_k) = w_k$$

has a solution $\forall \{w_k\}_{k=1}^\infty \in \ell^\infty$ are called <u>interpolating</u>.

<u>Remark</u>: 1) If $\{z_n\}$ is interpolating, then $\exists C = C(\{z_n\}_1^\infty)$ s.t.
$\forall \{w_n\} \in \ell^\infty \quad \|f\|_\infty \le C \|\{w_n\}^\infty\|_\infty$.

Pf: $T: H^\infty \to \ell^\infty$

$Tf = \{f(z_k)\}_1^\infty$.

$\{z_k\}$ interpolating $\Leftrightarrow T$ is surjective.

$\Rightarrow T$ open

$\Rightarrow T^{-1}: \ell^\infty \to H^\infty/\ker T$ is bounded.