Last time

$\{z_n\}$: interpolating iff $\forall \{w_n\}_1^\infty \in \ell^\infty$ $\exists f \in H^\infty$ $f(z_k) = w_k$ $\forall k$

$$\|f\|_\infty \le C \|\{w_n\}\|_\infty .$$

Necessary condition

If $\{z_k\}_1^\infty$ interpolating, then $\forall n$ $f(z_k) = \begin{cases} 0 & k \ne n \\ 1 & k = n. \end{cases}$

has a solution (let's denote f_n), $\|f_n\|_\infty < \infty$.

$$f_n = \prod_{k \ne n} b_{z_k} \cdot g_n \quad \Rightarrow \quad g_n(z_n) = \Big[\prod_{k \ne n} b_{z_k}(z_n) \Big]^{-1} = \frac{1}{B_{z_n}(z_n)} .$$

Notation. $B = \prod b_{z_k}$.

$B_n = B_{z_n} = B / b_{z_n} = \prod_{k \ne n} b_{z_k}$.

By max modulus, $\|g_n\|_\infty \ge \dfrac{1}{|B_{z_n}(z_n)|}$.

So, $\{z_n\}$ interpolating $\Rightarrow |B_n(z_n)| \ge \delta > 0$ $\forall n$ (CI). $\rightarrow$ (Carleson interpolating condition)

Theorem. (Carleson interpolation theorem)

$\{z_n\}_1^\infty$ is interpolating $\Leftrightarrow$ (CI)

pf. ($\Rightarrow$) Above↑

($\Leftarrow$) Restated. "$\forall \delta \in (0,1)$ $\exists C(\delta)$ s.t. $\forall \{z_n\}_1^\infty \in (\mathrm{CI})_\delta$ $\forall \{w_n\}_1^\infty \in \ell^\infty$

$\exists f \in H^\infty$ s.t. $f(z_n) = w_n$ $\forall n$ & $\|f\|_\infty \le C(\delta) \|\{w_n\}_1^\infty\|_\infty$."

Remark. Sufficient to prove thm for finite sequences $\{z_n\}_1^N$.

"Using standard normed families reasoning"

- $|f_n(z)| \le C < \infty$ on Ω

  $\exists f_{n_k} \to f$ uniformly on compact subsets of Ω.

- If $\forall K \subset \Omega$, $|f_n(z)| \le C_K < \infty$ $\forall z \in K$,

  then $\exists f_{n_k} \to f$ uniformly on compact subsets

$\cdot\ f_N : f_N(z_k) = w_k \quad 1 \le k \le N.$

If $\{z_n\}_1^\infty \in (CI)_\delta$, then $\{z_n\}_1^N \in (CI)_\delta$.

$\Rightarrow \exists f_N \ \ \|f_N\|_\infty \le C(\delta).$

So, $f_{N_k} \to f$.

$\Rightarrow \begin{cases} f(z_k) = w_k \quad \forall k. \quad (\overset{\text{because}}{f_N(z_k) = w_k} \text{ if } N > k). \\ \|f\|_\infty \le C(\delta). \end{cases}$ //

Assume $\{z_n\}$ – finite sequence.

$\|\{w_n\}\| = 1$. (Normalize)

$f_0(z_k) = w_k \quad \forall k.$

$f = f_0 - Bg$, $g \in H^\infty$ (general solution of interpolation problem)

$$\inf_{g \in H^\infty} \|f_0 - Bg\|_\infty = \inf_{g \in H^\infty} \left\| \frac{f_0}{B} - g \right\|_{L^\infty(\mathbb{T})}$$

Define $L \in (H^1)^*$.

∵ analytic.

$$L(h) = \frac{1}{2\pi i} \int_{\mathbb{T}} \left(\frac{f_0}{B} + \cancel{g}\right) h(z)\, dz. \qquad h \in H^1.$$

$$\left(|dz| = \frac{dz}{iz}, \quad \frac{dz}{i} = z\,|dz| \right)$$

<u>Goal</u>: $\|L\| \le C(\delta)$.

If we prove this, extend L to L' (by Hahn–Banach thm).

$\exists \varphi \in L^\infty \quad \|\varphi\|_\infty = \|L\| \le C(\delta)$

$$\int_{\mathbb{T}} \varphi h\, dz = \int_{\mathbb{T}} \frac{f_0}{B} h\, dz \quad \forall h \in H^1.$$

$\Rightarrow \frac{f_0}{B} - \varphi = g \in H^\infty$ (because take $h = z^n$, $n \ge 0$ and consider Fourier coefficients)

$\varphi B = f_0 + Bg \in H^\infty$

$\|\varphi B\|_{L^\infty(\mathbb{T})} = \|\varphi\|_{L^\infty(\mathbb{T})} \le C(\delta)$

$\|f_0 + Bg\|_\infty \le C(\delta)$,

$f_0 + Bg$ solves interpolation problem.

WLOG assume h is polynomial (since it is dense)

$$L(h) = \sum \mathrm{Res}_{z_k} \frac{f_0 h}{B} = \sum_k \underbrace{f_0(z_k)}_{w_k} h(z_k) \left(\frac{-z_k}{|z_k|}\right)(1 - |z_k|^2) \cdot \frac{1}{B_{z_k}(z_k)}.$$

$(\|\{w_k\}\|_\infty \le 1.)$

because
$$\left(\begin{array}{l} \operatorname{res}_a \dfrac{f(z)}{z-a} = f(a). \\ \\ B(z) = \dfrac{-|z_k|}{z_k} \cdot \dfrac{z-z_k}{1-\bar{z}_k z} \cdot B_{z_k} \end{array} \right).$$

By (CI).

$$\Rightarrow |L(h)| \leq \sum_k |h(z_k)| \cdot (1-|z_k|^2) \cdot \frac{1}{\delta}$$

$$= \frac{1}{\delta} \int_{\mathbb{D}} |h(z)|\, d\mu(z) \qquad \left(\mu = \sum (1-|z_k|^2)\, \delta_{z_k}\right)$$

$$\overset{?}{\leq} C \|h\|_1$$

<u>Def</u>. μ in $\mathbb{D}$ is called p-Carleson $1 \leq p < \infty$

if $\exists C < \infty$ $\forall h \in H^p$

$$\int_{\mathbb{D}} |f|^p \, d\mu(z) \leq C \int_{\mathbb{T}} |f(z)|^p \frac{|dz|}{2\pi} = C \|f\|_p^p$$

$H^p \subset L^p(\mu)$.