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Recall:

p-Carleson measure μ if $\forall h \in H^p$

$$(*) \quad \int_{\mathbb{D}} |h(z)|^p \, d\mu(z) \le C \int_{\mathbb{T}} |h(z)|^p \frac{|dz|}{2\pi} \qquad C \text{ depends only on } \mu$$

$$H^p \subset L^p(\mu)$$

Rk: μ is p-Carleson iff $(*)$ holds for all outer $h \in H^p$

Rk: μ is p-Carleson and $f \in H^{p_1}$ outer

$\Rightarrow f^{p_1/p} \in H^p$.

Note: f has no zeros in $\mathbb{D}$

since p-Carleson

So
$$\int_{\mathbb{D}} |f|^{p_1} \, d\mu = \int |f^{p_1/p}|^p \, d\mu \le C \int |f^{p_1/p}|^p \frac{|dz|}{2\pi} = C \int_{\mathbb{T}} |f|^{p_1} \frac{|dz|}{2\pi}$$

So any p-Carleson measure is p_1-Carleson with the same constant C.

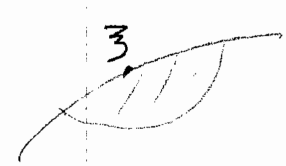
In particular,

1-Carl $\Leftrightarrow$ 2-Carl.

Thm (Carleson Embedding Thm). If $\mu \ge 0$ is a measure on $\mathbb{D}$, TFAE:

(1) $$\int_{\mathbb{D}} |f|^2 \, d\mu \le C_1 \int_{\mathbb{T}} |f|^2 \frac{|dz|}{2\pi} \qquad \forall f \in H^2$$

(2) $$\int_{\mathbb{D}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda} z|^2} \, d\mu(z) \le C_2 \qquad \forall \lambda \in \mathbb{D}$$



($\tilde{2}$) " " $\le \tilde{C}_2 \qquad \forall \lambda \in (\operatorname{supp} \mu) \cap \mathbb{D}$

(3) $\forall \xi \in \mathbb{T}$ $\forall r > 0$, $\mu\{z \in \mathbb{D} \mid |z - \xi| < r\} \le C_3 r$.

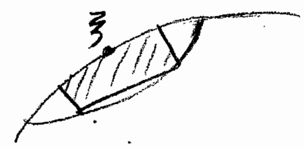
Moreover, given μ, the best possible constants $C_1, C_2, \tilde{C}_2, C_3$ are equivalent, in the sense that

$$A \asymp B \quad \exists K \text{ s.t. } \frac{1}{K} A \le B \le K A.$$

Rk: In (3), we can use any

$$Q_I = \Big\{ z \in \mathbb{D} \;\Big|\; \frac{z}{|z|} \in I,\ 1 - \underbrace{|I|}_{\text{length of } I} \le |z| \le 1 \Big\}$$

where I is an arc. For example,



$$\underbrace{(CI)}_{\text{Carleson interpolation thm}} \Rightarrow \tilde{2} \quad \text{for } \mu = \sum (1 - |z_k|^2)\delta_{z_k}$$

"Magic" formula:

$$(*) \quad 1 - \left| \frac{z - \lambda}{1 - \bar{\lambda} z} \right|^2 = \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda} z|^2} \qquad \text{(proof: exercise)}$$

$$\ln \delta \le \ln |B_{z_n}(z_n)| = \frac{1}{2} \sum_{k \ne n} \ln |b_{z_k}(z_n)|^2 \qquad \text{(Recall } \ln x \le x - 1\text{)}$$

$$\le \frac{1}{2} \sum_{k \ne n} \left(1 - |b_{z_k}(z_n)|^2\right) \underset{\text{by } (*)}{=} -\frac{1}{2} \sum_{k \ne n} \frac{(1 - |z_k|^2)(1 - |z_n|^2)}{|1 - \bar{z}_n z_k|^2}$$

$$\sum_{k:\, k \ne n} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z_k|^2} (1 - |z_k|^2) \le -2 \ln \delta = 2 \ln \frac{1}{\delta},$$

where δ is from the Carleson interpolation condition:

$$|B_{z_n}(z_n)| \ge \delta > 0.$$

and $$\sum_k \frac{1-|z_n|^2}{|1-\bar z_n z_k|^2}(1-|z_k|^2) \le 2\ln\frac{1}{\delta}+1.$$

Geometric description of interpolating sequences

Rk: $|B_{z_n}(z_n)| \ge \delta \quad \Rightarrow \quad |b_{z_k}(z_n)| \ge \delta \quad \forall n,k \quad n\neq k.$

What does it mean geometrically?

The points z_k are separated in the hyperbolic metric.

(Pseudohyperbolic distance cannot be less than δ.)

Prop: If $|b_{z_k}(z_k)| \ge \delta > 0$ and $\sum (1-|z_k|^2)\delta_{z_k}$ - Carleson

then $\{z_n\} \in (CI)$ ⇝ Carleson interpolation sequences

We use the following.

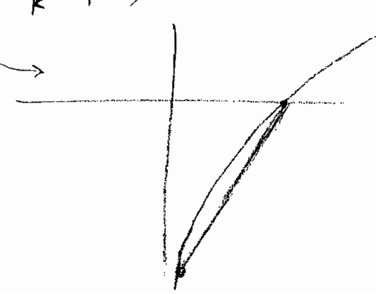
Lemma Let z, z_k, $k \ge 1$ s.t. $|b_{z_k}(z)| \ge \delta > 0$, and

let $$\sum_k \frac{1-|z|^2}{|1-\bar z z_k|^2}(1-|z_k|^2) \le C.$$ ⇝ "Carleson potential"

Then $\ln|B(z)| \ge \frac{1}{2}\ln\delta \cdot C$

Pf $$\ln|B(z)| = \frac{1}{2}\sum_k \ln|b_{z_k}(z)|^2 \ge \frac{1}{2}(\ln\delta)\sum_k(1-|b_{z_k}(z)|^2)$$

(by the inequality $\ln x \ge (\ln\delta)(1-x)$ for $\delta \le x \le 1$) ⟶



$$= \frac{1}{2}\ln\delta \sum \frac{(1-|z|^2)(1-|z_k|^2)}{|1-\bar z z_k|^2}. \quad \square$$

Pf of proposition

Applying this lemme to $B = B_{z_n}$ taking $z = z_n$,

we get the statement of the proposition.

Pf of Thm (Carleson Embedding Thm)

① ⇒ ② If we consider $k_\lambda = \dfrac{1-|\lambda|^2}{1-\bar{\lambda} z}$ then $k_\lambda \in H^2$

$\|k_\lambda\|_2 = 1.$

② ⇒ $\tilde{②}$ Also, trivial.

② ⇒ ③

Given λ let $\zeta = \dfrac{\lambda}{|\lambda|}$, let $r = 1-|\lambda|^2$.

Then if $z \in \mathbb{D}$ s.t. $|z-\zeta| < r$, then

$$\frac{1-|\lambda|^2}{|1-\bar{\lambda}z|^2} \geq \frac{c}{r}$$

where c is some "absolute" constant

(depends only on geometry of the unit disk)

Integrating this over a small disk of radius r, we get the result.

③ ⇒ ② (Will be proved next lecture.)

$\tilde{②}$ ⇒ ① (Not trivial)