Proving CET

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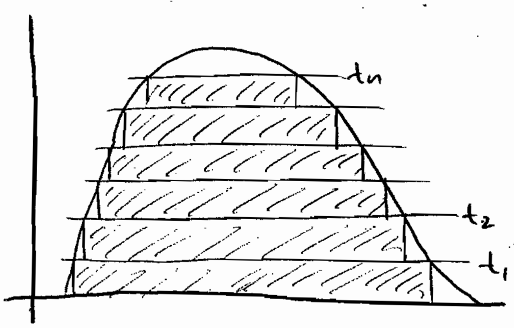
(3) ⇒ (2):

[Reminder: (3): $\mu\left(\{z \in \mathbb{D} : |\zeta - z| \leq r\}\right) \leq C_3 r \quad \forall \zeta \in \mathbb{T}, \quad \forall r > 0$

(2): $\displaystyle\int \frac{1-|\lambda|^2}{|1-\bar{\lambda} z|^2}\, d\mu(z) \leq C_2 \quad \forall \lambda \in \mathbb{D}$]

Fact from real analysis:

$$F \geq 0, \quad \int F\, d\mu = \int_0^\infty \mu\left(\{x : F(x) > t\}\right) dt$$



Lebesgue integral:
sorted by the values it takes
(like a bank teller sorting bills by value)

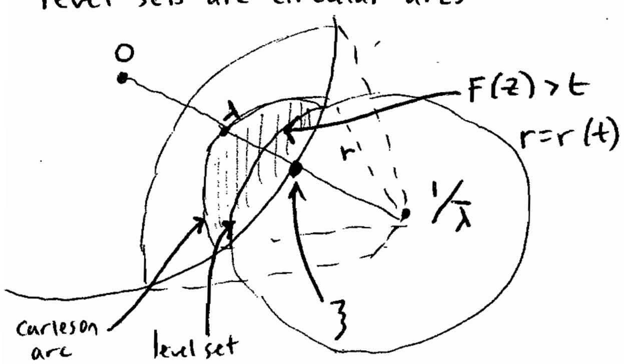
Area of rectangle: $\underset{\text{width}}{\mu(F > t_n)} \cdot \underset{\text{height}}{(t_n - t_{n-1})}$

want to find level sets

$$\frac{1-|\lambda|^2}{|1-\bar{\lambda} z|^2} > t \quad \Rightarrow \quad \frac{1-|\lambda|^2}{|\lambda|^2 \left|1/\bar{\lambda} - z\right|^2} > t$$

reflection of λ in the unit circle (actually inversion)

level sets are circular arcs



$$\mu\left(F(z) > t\right) \leq \mu\left\{z \in \mathbb{D} : |z - \zeta| < r(t)\right\}$$

$$\leq C r \quad \text{(condition (3))}$$

$$\leq \frac{1}{2} C \left| \left\{ z \in \mathbb{T} : \frac{1-|\lambda|^2}{|1-\bar{\lambda} z|^2} > t \right\} \right|$$

radius is bounded by half of arc

Lebesgue measure of this set "length"

so we get by integrating $\int_0^\infty \ldots dt$

$$\int \frac{1-|\lambda|^2}{|1-\bar{\lambda} z|^2}\, d\mu(z) \leq \frac{C}{2} \int \frac{1-|\lambda|^2}{|1-\bar{\lambda} z|^2}\, |dz| = \frac{C}{2}(2\pi) = \pi C$$ because integrating Poisson kernel

this proves that (2) ⇒ (3).

$(\tilde{2}) \Rightarrow (1)$: $\quad (\tilde{2}) \displaystyle\int \frac{1-|\lambda|^2}{|1-\bar{\lambda}z|^2}\, d\mu(z) \le C_2 \quad \forall\, \lambda \in \mathbb{D} \cap \operatorname{supp} \mu$

$(1) \displaystyle\int |f^2(z)|\, d\mu(z) \le C_1 \int |f(z)|^2 \frac{|dz|}{2\pi}$ so we have an embedding

Fun with Green's Formula

Stokes thm: $\displaystyle\int_{\partial\Omega} \omega = \int_{\Omega} d\omega$ $\quad\Omega$ can be a region with sufficiently smooth boundary, or a manifold, ...

Let $\underset{u,v}{\Omega} \subset \mathbb{C} = \mathbb{R}^2$

$\omega = u\,(v_x\, dy - v_y\, dx) - v\,(u_x\, dy - u_y\, dx)$ a 1-form

$= \left(u \dfrac{dv}{dn} - v \dfrac{du}{dn}\right) ds$

n : normal derivative

$d\omega = u\, \Delta v - v\, \Delta u$ (Laplace)

$\displaystyle\iint_{\Omega} (u\Delta v - v\Delta u)\, dx\, dy = \int_{\partial\Omega} \left(u \frac{dv}{dn} - v \frac{du}{dn}\right) ds$

n = outer unit normal; ds: integration w.r.t. arc length

assume $\partial\Omega$ is piecewise C^1 and $u, v \in C^2(\overline{\Omega})$

$u \in C^2(\mathbb{D})$, $\quad v = \ln \dfrac{1}{|z|} = -\ln|z|$

$\Omega_\varepsilon = \mathbb{D} \setminus D_{0,\varepsilon}$ $\quad\leftarrow$ disk minus a disk of radius ε centered at 0

$\displaystyle\iint_{\Omega_\varepsilon} (\Delta u) \ln \frac{1}{|z|} = \cancel{\int_{\mathbb{T}} u\, |dz|}$ $\qquad$ notice that $\displaystyle -\frac{d}{dr} \ln r \Big|_{r=1} = -1$

So we get a negative sign in front of $v\Delta u$ and $-v\dfrac{du}{dn}$.

$$= \int_{\mathbb{T}} u\, |dz| - \underbrace{\int_{|z|=\varepsilon} u(z) \cdot \frac{1}{\varepsilon}\, |dz|}_{\downarrow \text{ as } \varepsilon \to 0:\ 2\pi\, u(0)} + \underbrace{\int_{|z|=\varepsilon} \ln \frac{1}{|z|} \frac{\partial u}{\partial n}}_{\downarrow \text{ as } \varepsilon \to 0:\ 0}$$

we know $\left|\dfrac{\partial u}{\partial n}\right| \le |\nabla u|$ gradient is continuous so $\dfrac{\partial u}{\partial n}$ is bounded

$$\int u \frac{|dz|}{2\pi} - u(0) = \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta u \ln\frac{1}{|z|}\, dA(z) \qquad dA(z) = dx\,dy$$

(integration w.r.t. area)

Fix $z_0 \in \mathbb{D}$, u, $v = \ln\left|\frac{1-\bar{z}_0 z}{z-z_0}\right|$ function is zero on the boundary

Möbius transform

$$\Omega_\varepsilon = \mathbb{D} \setminus \mathbb{D}_{z_0,\varepsilon}$$

then $\mathcal{H} u(z_0) - u(z_0) = \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta u(z) \underbrace{\ln\left|\frac{1-\bar{z}_0 z}{z-z_0}\right|}\, dA(z)$

$\uparrow$ Harmonic extension of u from the boundary

Green's function: Harmonic except at pole z_0 boundary values of 1

$G_{\mathbb{D}}(z, z_0)$

note: $\frac{d}{dn}\ln\left|\frac{1-\bar{z}_0 z}{z-z_0}\right| = -\frac{1-|z_0|^2}{|1-\bar{z}_0 z|^2}$.

Poisson Kernel

subharmonic function: bounded by harmonic function

function with nonnegative Laplacian

$u \in$ bdd, $\Delta u \geq 0 \Rightarrow \Delta u \ln\frac{1}{|z|}\,dx\,dy$ Carleson

in fact, all Carleson measures are obtained this way