<u>Lemma</u>: If $u \in C^2(\Omega)$ and $\Delta u \geq 0$, then $\forall z_0 \in \Omega$ and $\forall r$ s.t. $\overline{B_{z_0,r}} \subset \Omega$,

then $\overset{(1)}{} \frac{1}{2\pi r}\int_{|z-z_0|=r} u(z)\,|dz| \geq u(z_0)$ and $\overset{(2)}{} \frac{1}{\pi r^2}\iint_{|z-z_0|<r} u(z)\,dA(z) \geq u(z_0)$.

<u>Pf</u>: (1) follows from Green's formula if $z_0 = 0$ and $r=1$.

$$\int_{\mathbb{T}} u(z)\frac{|dz|}{2\pi} - u(0) = \iint_{\mathbb{D}} \underbrace{\ln\frac{1}{|z|}\,\Delta u(z)}_{\geq 0}\,dA(z)$$

For general z_0, change coordinates.

(2) follows from (1) by integrating in polar coordinates.

Note: u with $\Delta u \geq 0$ is called a <u>subharmonic</u> function.

<u>Lemma</u> (Uchiyama's lemma)

If: $\varphi \in C^2(\mathbb{D})$, $\varphi \leq 0$, $\Delta\varphi \geq 0$, let $\nu = \frac{1}{2\pi}e^{\varphi}\Delta\varphi \ln\frac{1}{|z|}\,dA(z)$ (where $dA(z) = dx\,dy$).

Then $\forall f \in H^2$, $\int_{\mathbb{D}} |f(z)|^2\,d\nu(z) \leq \|f\|_2^2$

<u>Corollary</u>: $\varphi \in C^2(\mathbb{D})$, $-k \leq \varphi \leq 0$, $\Delta\varphi \geq 0$. Then $\forall f \in H^2$, $\frac{1}{2\pi}\int_{\mathbb{D}} |f(z)|^2 \Delta\varphi \ln\frac{1}{|z|}\,dA(z) \leq ek\|f\|_2^2$

Pf of corollary: $-k \leq \varphi \leq 0$

$-1 \leq \frac{\varphi}{k} \leq 0$

Applying the lemma, $\frac{1}{2\pi}\iint_{\mathbb{D}} |f|^2 \frac{\Delta\varphi}{k}\ln\frac{1}{|z|}\,dA(z) \leq e\|f\|_2^2$

(use $e^{\varphi/k} \geq \frac{1}{e}$).

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ Sometimes, we'll write ∂.

$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ $\bar{\partial}$.

If $f \in \text{Hol}$, $\partial f = f'$, $\boxed{\bar{\partial} f = 0}$ — Cauchy-Riemann equations.

$\partial \bar{f} = 0$, $\bar{\partial}\bar{f} = \overline{f'}$

$$\partial\bar{\partial} = \bar{\partial}\partial = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \frac{1}{4}\Delta.$$

Proof of Uchiyama's Lemma:

$\Delta e^{\varphi}|f|^2 = ?$

$\partial(e^{\varphi}|f|^2) = e^{\varphi}\partial\varphi\cdot|f|^2 + e^{\varphi}f'\bar{f}$ (since $\partial(f\bar{f}) = (\partial f)\cdot\bar{f} + f\,\overset{0}{\partial\bar{f}}$).

$\bar\partial\partial(e^{\varphi}|f|^2) = e^{\varphi}\bar\partial\varphi\,\partial\varphi\,|f|^2 + e^{\varphi}\partial\varphi\, f\overline{f'} + e^{\varphi}\bar\partial\partial\varphi\,|f|^2 + e^{\varphi}\bar\partial\varphi\, f'\bar{f} + e^{\varphi}f'\overline{f'}$

$= e^{\varphi}\bar\partial\partial\varphi\,|f|^2 + \underbrace{e^{\varphi}|\partial\varphi f + f'|^2}_{\geq 0}$ (used $\bar\partial\varphi = \overline{\partial\varphi}$).

$\therefore\ \Delta e^{\varphi}|f|^2 \geq e^{\varphi}\Delta\varphi\,|f|^2 \geq 0.$

So $\frac{1}{2\pi}\iint_{\mathbb{D}} e^{\varphi}\Delta\varphi\,|f|^2 \ln\frac{1}{|z|}\,dA(z) \leq \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta(e^{\varphi}|f|^2)\ln\frac{1}{|z|}\,dA(z)$

$= \int_{\mathbb{T}} e^{\varphi}|f|^2\frac{|dz|}{2\pi} - e^{\varphi(0)}|f(0)|^2$

$\leq \int_{\mathbb{T}}|f|^2\frac{|dz|}{2\pi}.$

(Really should do this for $f(rz)$, $\varphi(rz)$, $r<1$, then take the limit as $r\to 1$).

Theorem. μ measure on $\mathbb{D}$, $\mu\geq 0$

$\int_{\mathbb{D}}\frac{1-|\lambda|^2}{|1-\bar\lambda z|^2}\,d\mu(z) \leq K < \infty$. Then $\forall f\in H^2$, $\int_{\mathbb{D}}|f(z)|^2 d\mu(z) \leq 2eK\|f\|_2^2$

(This is $\tilde{2}\Rightarrow 1$).

Pf (S. Petermichl, S. Treil, B. Wick).

WLOG, $K=1$. and $\operatorname{supp}\mu\subset D_{0,r}$.

$\varphi(z) = -\int\frac{1-|z|^2}{|1-\bar\lambda z|^2}\,d\mu(\lambda)$. Note that $-1\leq\varphi\leq 0$.

$\Delta\varphi = \int_{\mathbb{D}}\Delta\left(\frac{|z|^2-1}{|1-\bar\lambda z|^2}\right)d\mu(\lambda)$.

If $f\in\mathrm{Hol}$, $\Delta|f|^2 = 4\partial\bar\partial f\bar{f} = 4f'\overline{f'} = 4|f'|^2$.

So $\Delta\varphi = \int_{\mathbb{D}}\Delta(\sim) = \int_{\mathbb{D}}\Delta\left(\left|\frac{z}{1-\bar\lambda z}\right|^2 - \left|\frac{1}{1-\bar\lambda z}\right|^2\right) = 4\int_{\mathbb{D}}\left|\left(\frac{z}{1-\bar\lambda z}\right)'\right|^2 - \left|\left(\frac{1}{1-\bar\lambda z}\right)'\right|^2 = \int_{\mathbb{D}} 4\underbrace{\frac{1-|\lambda|^2}{|1-\bar\lambda z|^4}}_{\geq 0}.$

Set $d\nu = \frac{1}{2\pi}e^{\varphi(z)}\Delta\varphi(z)\ln\frac{1}{|z|}\,dA(z)$.

By Uchiyama's lemma, $\int|f|^2 d\nu \leq \|f\|_2^2\quad\forall f\in H^2$.

$\int_{D}|f|^2 d\mu \overset{?}{\leq} 2e\int_{\mathbb{D}}|f|^2 d\nu$?

$\int_{\mathbb{D}}|f|^2 d\nu = \frac{4}{2\pi}\underbrace{\iint_{\mathbb{D},\mathbb{D}} e^{\varphi(z)}\frac{1-|\lambda|^2}{|1-\bar\lambda z|^4}|f|^2\ln\frac{1}{|z|}\,dA(z)\,d\mu(\lambda)}$

$\geq \frac{1}{2e}|f(\lambda)|^2\quad\forall\lambda\in\operatorname{supp}\mu$. (with the constant).

We know $\Delta(e^{\varphi}|f|^2)\geq 0$. Integrate only over $D_{\lambda,\frac{1-|\lambda|}{10}}$. $\ln\frac{1}{|z|}\geq\frac{1}{2}(1-|z|^2)$, so $\frac{(1-|\lambda|^2)(1-|z|^2)}{|1-\bar\lambda z|^4} \geq C\frac{1}{(1-|\lambda|)^2}$.