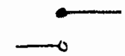
3/2/2009 Complex Analysis II

Subharmonic functions

Def: $v : \Omega \to [-\infty, \infty)$ is subharmonic if

1. v is upper-semicontinuous i.e. $\forall z_0 \in \Omega$ $v(z_0) \geq \limsup_{z \to z_0} v(z)$ e.g. 
2. $\forall z_0 \in \Omega$ $\exists r_0$ s.t. $D_{z_0, r_0} \subset \Omega$ s.t. $\forall r \leq r_0$ $v(z_0) \leq \frac{1}{\pi r^2} \iint_{|z-z_0|<r} v(z)\, dA(z)$

Sometimes the condition is formulated as $v(z_0) \leq \frac{1}{2\pi r} \int_{|z-z_0|} v(z)\, |dz|$.

It implies the first by integrating in polar coordinate. In fact they will be equivalent.

If f_α $\alpha \in A$ are continuous, then the function $f(x) = \inf_\alpha f_\alpha(x)$ is upper semi continuous.

In particular, if $f_k(x)$ decreases as $k \to \infty$. Then $\lim_{k \to \infty} f_k(x)$ is upper semi continuous.

If $f(x) = \inf_k f_k(x)$, construct $g_n(x) = \min\{f_1(x), \dots, f_n(x)\}$. Then $g_n(x)$ is decreasing and

$f(x) = \lim_{n \to \infty} g_n(x)$.

Remark: Any upper semicontinuous function on a compact metric space is the limit of a ptwise decreasing sequence of continuous functions.

Remark: Upper semicontinuous function on a compact set attains its max

Fact: $v \in SH(\Omega)$ (subharmonic). Suppose Ω open and connected.

If $z_0 \in \Omega$ is a local max $\Rightarrow$ $v \equiv$ const.

Prop: v USC, v is subharmonic in Ω if $\forall$ open bounded $W \subset \Omega$, $\forall u \in Harm(W)$

$\forall \xi \in \partial W$, $\limsup_{z \to \xi} (v(z) - u(z)) \leq 0$ $\Rightarrow$ $v(z) \leq u(z)$ $\forall z \in W$.

Pf: It's in the textbook.

Cor: If $v \in SH$, $\varphi \in \mathcal{H}ol$, then $v \circ \varphi$ is subharmonic.

Pf: Follows from the trivial fact: If u harmonic, φ $\mathcal{H}ol$, then $u \circ \varphi$ is harmonic. This is because $u = \mathrm{Re}\{F\}$, $F \in \mathcal{H}ol$ locally. So $u \circ \varphi = \mathrm{Re}\{F \circ \varphi\}$. Thus $u \circ \varphi$ harmonic, as $F \circ \varphi$ holomorphic.

Prop: Suppose $v \in C^2(\Omega)$. Then $v \in SH(\Omega) \iff \Delta v \geq 0$.

Pf: Exercise. Use Green's formula

Even if $v \notin C^2(\Omega)$, the proposition holds in the sense of distributions. (non-neg distributions are measure)

Theorem: (Riesz representation theorem for subharmonic functions).

Let $f \in SH(\Omega)$. $\exists! \, \mu \geq 0$ in Ω s.t. $\forall W$ open bounded w/ $\overline{W} \subset \Omega$,

$$f(z) = -\int_W \ln \frac{1}{|\xi - z|} \, d\mu(\xi) + h_W(z)$$ where h_W is harmonic. μ is called the Riesz measure.

Formally, $\Delta f = \mu$.