Examples of subharmonic functions.

1. If $f \in Hol(\Omega)$, then $\ln|f| \in SH(\Omega)$:

   - If $f(z_0) \neq 0$, then $\ln|f|$ is harmonic in a neighborhood of z_0.

     $\Rightarrow \ln|f(z_0)| = \frac{1}{\pi r^2} \iint_{|z-z_0|<r} \ln|f(z)|\, dA(z)$;

   - If $f(z_0) = 0$,

     $-\infty = \ln|f(z_0)| \leq \frac{1}{\pi r^2} \iint_{|z-z_0|<r} \ln|f(z)|\, dA(z)$.

2. If $f \in Harm(\Omega)$, $p \geq 1$, then $|f|^p$ is SH:

   - $|f(z_0)|^p = \left|\int_{|z-z_0|=r} f(z) \frac{|dz|}{2\pi r}\right|^p \leq \left(\int_{|z-z_0|=r} |f| \frac{|dz|}{2\pi r}\right)^p \leq \int_{|z-z_0|=r} |f|^p \frac{|dz|}{2\pi} \cdot \cancel{\left(\int_{|z-z_0|=r} 1^{p'} \frac{|dz|}{2\pi}\right)^{\frac{p}{p'}}}^{\,1}$.

3. If $f \in Hol(\Omega)$ $p > 0$, then $|f|^p \in SH$:

   __Lemma.__ If $v \in SH(\Omega)$, $\varphi : [-\infty, \infty) \to [-\infty, \infty)$, $\varphi \uparrow$

   convex, continuous at $-\infty$,

   then $\varphi \circ v \in SH(\Omega)$.

   So, if we apply this lemma to $v = \ln|f|$ and $\varphi(t) = e^t$, then we get $|f|^p \in SH$.

   __Lemma__ (Jensen's inequality)

   $X, \mu, \mu(X) = 1$. $\quad v : X \to \mathbb{R}$.

   $\varphi : \mathbb{R} \to \mathbb{R}$ convex.

   $\Rightarrow \varphi\left(\int_X v\, d\mu\right) \leq \int_X \varphi(v)\, d\mu$.

Harmonic majorant & conformal invariance.

def of H^p.

$v \in SH(\Omega)$

$\exists u$ s.t. $u \in Harm(\Omega)$ $v \leq u$.

__Prop.__ $v \in SH(\Omega)$, v has a harmonic majorant.

$\Rightarrow \exists!$ least harmonic majorant of v.

__sketch of the proof.__ $u(z) = \inf\{f(z) : -f \text{ subharmonic}, f \geq v\}$

$= -\sup\{g(z) : g \text{ subharmonic}, g \leq -v\}$ $\quad \big)\ g = -f$.

Clearly, if $U \in Harm(\Omega)$ & $v \le U$, then $u \le U$. $\forall z \in \Omega$.

Want to show that u is harmonic.

<u>Lemma</u>. $g_\alpha \in SH(\Omega)$. $G(z) = \sup_\alpha g_\alpha(z)$.

$\Rightarrow \forall z_0 \in \Omega \ \forall r$ s.t. $D(z_0,r) \subset \Omega$

$$G(z_0) = \int_{|z-z_0|=r} G(z) \frac{|dz|}{2\pi r}.$$

<u>pf</u>. Exercise.

<u>Lemma</u>. $G(z) = \sup\{g(z) : g(z) \in SH(\Omega),\ g \le -v\}$.

$$\forall D(z_0,r) \subset \Omega,\quad \tilde{G}(z) = \begin{cases} G(z) & z \notin D(z_0,r) \\ \text{Harmonic extension of } G|_{|z-z_0|=r} \text{ to } D(z_0,r). \end{cases}$$

$\Rightarrow \tilde{G}(z) \le -v$.

(Easy to check $\tilde{G} \in SH(\Omega)$.).

($\Rightarrow \tilde{G} = G$.)

see ex. 2 in HW assignment 2 for details

<u>This is "Perron process".</u>

<u>Ex</u>. $\Omega = \mathbb{D}$.

$v \in SH(\Omega)$.

u_r on $D_r = \{z : |z| < r\}$.

$\uparrow$ Harmonic extension of $v|_{|z|=r}$ by Poisson integral formula.

$z \in \mathbb{D}$. $u_r(z) \overset{r\to 1}{\nearrow} u(z)$ $r > |z|$.

$\Rightarrow u(z)$ - harmonic or $\equiv \infty$

(Why? $z_0 \in \mathbb{D}$. $\varepsilon > 0$. $\overline{D(z_0,\varepsilon)} \subset \mathbb{D}$

$\Rightarrow \overline{D(z_0,\varepsilon)} \subset D_r$ $\forall r > r_0$. $(r_0 < 1)$,

$$u_r(z_0) = \int_{|z-z_0|=\varepsilon} u_r(z) \frac{|dz|}{2\pi\varepsilon}$$

$r \to 1 \ \downarrow \qquad \downarrow$ Monotone Convergence Theorem.

$$u(z_0) = \int_{|z-z_0|=\varepsilon} u(z) \frac{|dz|}{2\pi\varepsilon}.$$

Def. Ω

$H^p(\Omega) = \{ f \in Hol(\Omega)$ s.t. $|f|^p$ has harmonic majorant $\}$.

Fix $z_0 \in \Omega$. $\|f\|_{H^p}^p = u(z_0)$. ... Norm of $H^p(\Omega)$. (Diffent choice of $z_0 \Rightarrow$ diffent norm
$\Rightarrow$ But equivalent)

If $\Omega = \mathbb{D}$, $z_0 = 0$, then $H^p(\Omega)$ is a classical H^p space.