Recall last time:

$H^p(\Omega) = \{ f \in Hol(\Omega) \mid |f|^p \text{ has harmonic majorant} \}$

Fix $\lambda \in \Omega$.

Then $\|f\|_{H^p}^p = u(\lambda)$ where u is the least harmonic majorant for $|f|^p$.

Why different choices of λ give equivalent norms:

Lemma (Harnak inequality).

Let $u \geq 0$, $u \in Harm(\mathbb{D})$, $r < 1$

Then $\forall z$ with $|z| < r$, $\frac{1-r}{1+r} u(0) \leq u(z) \leq \frac{1+r}{1-r} u(0)$.

Note: If we rescale the unit disc $\mathbb{D}$ to D_R, the constants become

$$\frac{R-r}{R+r}, \quad \frac{R+r}{R-r}.$$

Pf: Assume u is continuous in the closed unit disc.

$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\xi|^2} u(\xi) \frac{|d\xi|}{2\pi}$ and let $\rho = |z|$.

$$\frac{1-|z|^2}{|1-\bar{z}\xi|^2} \leq \frac{1-\rho^2}{(1-\rho)^2} = \frac{1+\rho}{1-\rho} \leq \frac{1+r}{1-r}$$

and $$\frac{1-|z|^2}{|1-\bar{z}\xi|^2} \geq \frac{1-\rho^2}{(1+\rho)^2} = \frac{1-\rho}{1+\rho} \geq \frac{1-r}{1+r}.$$

Any $u \in Harm(\mathbb{D})$ is continuous in $\overline{D_R}$ if $R < 1$.

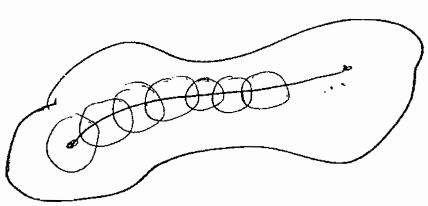
Thus, $\frac{R-r}{R+r} u(0) \leq u(z) \leq \frac{R+r}{R-r} u(0)$. Take the limit as $R \to 1$

to get $\frac{1-r}{1+r} u(0) \leq u(z) \leq \frac{1+r}{1-r} u(0)$.

Corollary: Ω a domain (open & connected). $z_1, z_2 \in \Omega$.

$u \geq 0$ and harmonic. Then $\exists\, c, C$ (depending only on Ω, z_1, z_2)

s.t. $c\,u(z_1) \leq u(z_2) \leq C\,u(z_1)$.



Use chain.

Harmonic measure

1. Dirichlet Problem: Ω a domain, $\Omega \subset \hat{\mathbb{C}}$ (Riemann sphere).

Given $f \in C(\partial\Omega)$ find $u \in C(\mathrm{cl}\,\Omega) \cap \mathrm{Harm}(\Omega)$ s.t. $u|_{\partial\Omega} = f$.

This is possible if $\partial\Omega$ does not have trivial (single point) connected components.

Proposition: u harmonic, bounded on $\{z \mid 0 < |z - z_0| < \varepsilon\}$. Then u can be extended to a harmonic function on $\{z \mid |z - z_0| < \varepsilon\}$.

Perron Process: $\mathcal{F} = \{v \mid v \in SH(\Omega),\ \forall \xi \in \partial\Omega,\ \limsup_{z \to \xi} v(z) \le f(\xi)\}$.

$\mathcal{F}$ satisfies: 1) $v_1, v_2 \in \mathcal{F} \Rightarrow \max(v_1, v_2) \in \mathcal{F}$

2) $v \in \mathcal{F} \Rightarrow \forall D \subset \Omega$, $\tilde{v} = \begin{cases} v(z) & \text{if } z \notin D \\ \text{Harmonic ext. of } v|_{\partial D} \text{ at } z & \text{if } z \in D. \end{cases}$

$\tilde{v} \in \mathcal{F}$.

$u(z) = \sup\{v(z) \mid v \in \mathcal{F}\}$

u harmonic or $u = \infty$.

Harmonic Measure

Ω, $\lambda \in \Omega$.

Consider the functional

$C(\partial\Omega) \ni f \longmapsto u(\lambda)$, where u is the solution of the Dirichlet problem.

Properties: $f \ge 0 \Rightarrow u \ge 0 \Rightarrow u(\lambda) \ge 0$.

$f = 1 \mapsto 1$.

Positive linear functionals on $C_0(X)$ are Radon measures.

If K is compact and Hausdorff, any finite Borel measure on K is a Radon measure.

Conclude that $\exists!$ Borel measure ω_λ on $\partial\Omega$ s.t. $u(\lambda) = \int_{\partial\Omega} f(z)\, d\omega_\lambda(z)$.

We have $c\,\omega_{\lambda_1} \le \omega_{\lambda_2} \le C\,\omega_{\lambda_1}$, so changing the point λ gives equivalent measures.

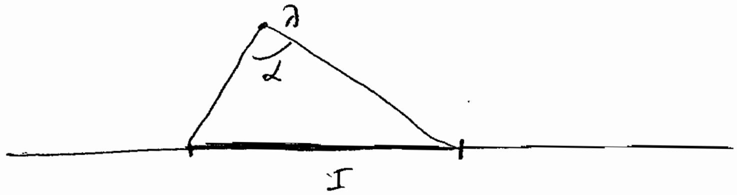
Examples:

1. $\Omega = \mathbb{D}$. $d\omega_\lambda(z) = \dfrac{1 - |\lambda|^2}{|1 - \bar{\lambda} z|^2} \dfrac{|dz|}{2\pi}$.
2. $\Omega = \mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Im}\, z > 0\}$.

$d\omega_\lambda(x) = \dfrac{1}{\pi} \dfrac{\mathrm{Im}\,\lambda}{|x - \bar{\lambda}|^2}\, dx$. (Poisson kernel for $\mathbb{H}$).

②

Harmonic measure on the real line:



$$\omega_\lambda(I) = \frac{\alpha}{\pi}.$$

Probabilistic interpretation of ω_λ: Ω.

$E \subset \partial\Omega$.

$\omega_\lambda(E) =$ Probability that Brownian motion started at λ ends up at E.