Properties of harmonic measure

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1. ω_λ is conformally invariant

$$\Omega_1 \xrightarrow{\varphi} \Omega_2$$

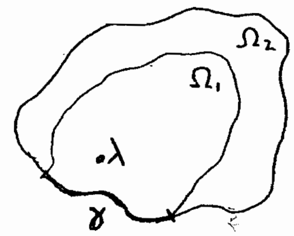
φ is a conformal mapping, so Ω_1, Ω_2 are conformally invariant
φ extends to the boundary, so gives a homomorphism between closures and boundaries

ω_1 ω_2 harmonic measures

$\omega^1_\lambda(E) = \omega^2_{\varphi(\lambda)}(\varphi(E))$ where E is a measurable subset $E \subset \partial\Omega_1$

we know that harmonic functions are conformally invariant, so there is nothing to prove.

2. Monotonicity



$\Omega_1 \subset \Omega_2$, let $\gamma = \partial\Omega_1 \cap \partial\Omega_2$
$\lambda \in \Omega_1$

$$\omega^1\big|_\gamma \le \omega^2\big|_\gamma$$

$f \in C(\partial\Omega_1)$ and supported on γ, $f \geq 0$

Solve Dirichlet problem: $\begin{cases} \Delta u_1 = 0 \\ u_1|_\gamma = f \\ u_1|_{\partial\Omega_1 \setminus \gamma} = 0 \end{cases}$ $\qquad \begin{cases} \Delta u_2 = 0 \\ u_2|_\gamma = f \\ u_2|_{\partial\Omega_2 \setminus \gamma} = 0 \end{cases}$

u_1 is only defined on Ω_1

$u_2|_{\Omega_1}$ solves: $\begin{cases} \Delta u = 0 \\ u|_\gamma = f \\ u|_{\partial\Omega_1 \setminus \gamma} = u_2|_{\partial\Omega_1 \setminus \gamma} \geq 0 \end{cases}$

we conclude $u_2 \geq u_1$ on Ω_1 (we never get equality except in the trivial case)

3. Equivalence of Harmonic measures

$\forall K \subset \Omega$, exists $C = C(K, \Omega)$ such that for all $\lambda_1, \lambda_2 \in K$,
compact

$$\frac{1}{C}\,\omega_{\lambda_1} \le \omega_{\lambda_2} \le C\,\omega_{\lambda_1}$$

Follows from Harnak.

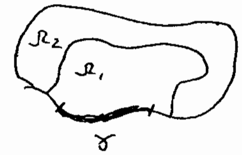
$\exists\, C = C(K, \Omega)$ such that for all $u \geq 0$, $\forall\, \lambda_1, \lambda_2 \in K$, $\Delta u = 0$ on Ω,

$$\frac{1}{C}\,u(\lambda_1) \le u(\lambda_2) \le C\,u(\lambda_1)$$

(Cover compact set by disks, apply Harnak's inequality for a disk.)

4. Localization

Now take γ to be strictly inside $\partial\Omega_1 \cap \partial\Omega_2$:



$\Omega_1 \subset \Omega_2$

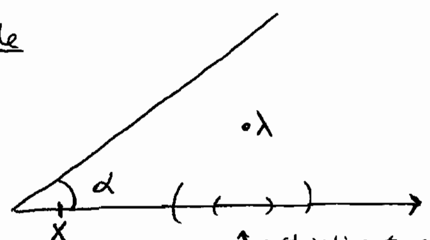
$\gamma \subset\subset \partial\Omega_1 \cap \partial\Omega_2$: closure of γ is inside the intersection; intersection is a nbd of γ.

$$\exists\, C \quad \text{s.t.} \quad \frac{1}{C}\,\omega^1_\lambda\Big|_\gamma \le \omega^2_\lambda\Big|_\gamma \le C\,\omega^1_\lambda$$

constant depending only on geometry

We don't care what happens outside a neighborhood of γ; then the proof is a combination of monotonicity and equivalence.

<u>Example</u>



cone

restricting to an interval, is essentrally Lebesgue measure
what is interesting is what happens near the angle

$$\omega^C_\lambda(0,x) = \omega^{\mathbb{C}_+}_{\lambda^p}(0,x^p) \approx x^{\pi/\alpha}$$

so if $\alpha = \frac{\pi}{2}$, measure of $(0,x) = x^2$
so measure is much smaller for small x

$p = \frac{\pi}{\alpha}$, power that we need to get to the halfplane

if $\alpha > \pi$, measure is bigger than x for small x.

We know how harmonic measure behaves on smooth curves and when we have cusps.
Harmonic measure is essentially a local property.

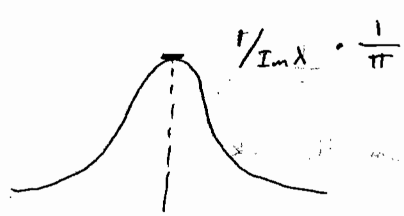
<u>Hardy spaces in $\mathbb{C}_+$</u>

$\mathbb{C}_+ = \{z : \operatorname{Im} z > 0\}$

<u>Def</u> $f \in H^p(\mathbb{C}_+)$ if $\displaystyle \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p\,dx < \infty$

$$\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p\,dx = \|f\|^p_{H^p}.$$

$$d\omega^{\mathbb{C}_+}_\lambda(x) = \frac{1}{\pi}\,\frac{\operatorname{Im}\lambda}{|x-\bar\lambda|^2}\,dx$$ basically just Poisson kernel



$\lambda = iR$, $R \to \infty$ then measure goes weakly to 0 because density $\to 0$

but $R\, d\omega^{\mathbb{C}_+}_{iR} \xrightarrow[R \to +\infty]{} \frac{1}{\pi}\,dx$ a multiple of Lebesgue measure

Connection with $H^p(\mathbb{D})$

$f \in \mathrm{Hol}(\mathbb{D})$ analytic on the unit disk

$$F(z) = \frac{\pi^{-1/p}}{(z+i)^{2/p}}\, f\left(\frac{z-i}{z+i}\right) \quad \text{for } z \in \mathbb{C}_+$$

<u>Thm</u> $f \longmapsto F$ is an isometric isomorphism $H^p(\mathbb{D}) \longrightarrow H^p(\mathbb{C}_+)$

(in 1-to-1 correspondence and preserves the norm)

$F = \mathcal{U} f$

1. $\mathcal{U}$ is an isometric isomorphism $L^p(\mathbb{T}) \longrightarrow L^p(\mathbb{R})$

   $f \in L^p(\mathbb{T})$

   then $\|f\|^p = \int_{\mathbb{T}} |f(w)|^p \frac{|dw|}{2\pi}$ change of variables $w = \frac{z-i}{z+i}$

   $$= \int_{\mathbb{R}} \left|f\left(\tfrac{z-i}{z+i}\right)\right|^p |w'(z)| \frac{|dz|}{2\pi} \quad \text{and } w = 1 - \frac{2}{z+i} \text{ so } |w'(z)| = \left|\frac{2}{z+i}\right|^2$$

   $$= \int_{\mathbb{R}} \left|f\left(\tfrac{z-i}{z+i}\right)\right|^p \frac{1}{|z+i|^2} \frac{|dz|}{\pi} = \int_{\mathbb{R}} |F(z)|^p \, dz .$$

Same as integral over a circle intersecting the unit circle at 1.