3/11/2009 COMPLEX ANALYSIS II

HW#5: $Lh = \sum (1-|z_k|^2)^{1/2} h(z_k)$ extend to L^2? Do same scheme as H^∞.

In H^∞, $f_0 + Bg$ is general sol'n.

Consider $\underbrace{\int \left(\frac{f_0}{B} + g\right) h \, dz}_{= L(h)}$ What does this functional give you. $|L(h)| \le c \sum (1-|z_k|^2) \underbrace{w_k}_{= f_0(z_w)} h(z_k)$

Apply Cauchy Schwarz above $\le \left(\sum (1-|z_k|^2)|w_k|^2 \right)^{1/2} \left(\sum |h(z_k)|^2 (1-|z_k|^2) \right)^{1/2}$

Use Carleson measure estimate.

interpolation cond $\Rightarrow$ carleson.

Riesz Projection, Cauchy integral, Plemelj formula, Hilbert transform

$H^2(\mathbb{D})$ $\quad P_+ f = \sum_{k \ge 0} \hat{f}(k) z^k$ $\qquad f \in L^2(\mathbb{T})$

converges absolutely in $H^2(\mathbb{D})$ because they're bounded.

Lemma: If $f \in L^1(\mathbb{T})$ $z \in \mathbb{D}$

Then $(P_+ f)(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\xi)}{\xi - z} \, dz$ usual Cauchy formula.

inner product w/ reproducing kernel.
$= (f, K_z)$

Pf:

If $f(z) = z^n$ $n \ge 0$, then

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\xi^n}{\xi - z} \, d\xi = z^n$$

If $f(z) = z^{-n}$ $n \ge 1$, then $\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\xi^{-n}}{\xi - z} \, d\xi = \frac{1}{2\pi i} \int_{R\mathbb{T}} \frac{\xi^{-n}}{\xi - z} \, d\xi \to 0$ as $R \to \infty$

Everything is well defined on $\mathbb{T}$. $\square$

$\int \frac{f(\xi)}{\xi - z} \, d\xi \xrightarrow[z \to z_0 \in \mathbb{T}]{}$ we would like to say P.V. $\int \frac{f(\xi)}{\xi - z_0} \, d\xi$. This is wrong.

Theorem: (Plemelj - Sokhotsky formulas)

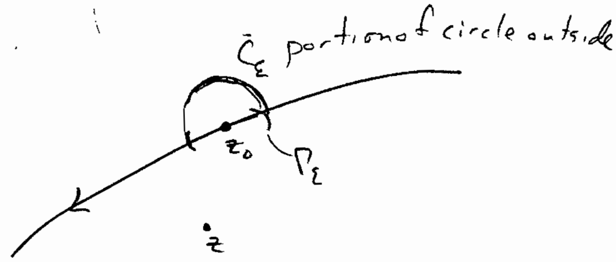
Ω bounded domain, $\Gamma = \partial \Omega$. Assume Γ is C^1 boundary. $f \in C^1(\Gamma)$

Define $F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi \quad z \in \mathbb{C} - \Gamma$

Then $\forall z_0 \in \Gamma \quad \lim\limits_{\substack{z \to z_0 \\ \text{non-tangentially}}} F(z) = \begin{cases} P.V. \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z_0} d\xi + \frac{1}{2} f(z_0) & \text{if } z \to z_0 \text{ from inside.} \\ P.V. \frac{1}{2\pi i} \int \frac{f(\xi)}{\xi - z_0} - \frac{1}{2} f(z_0) & \text{if } z \to z_0 \text{ from outside.} \end{cases}$

Pf: $z \in \Omega$

$\frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} + \frac{1}{2\pi i} \int_{\Gamma - \Gamma_\varepsilon}$



$= \underbrace{\frac{1}{2\pi i} \int_{\Gamma - \Gamma_\varepsilon}}_{} + \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{f(\xi) - f(z_0)}{\xi - z} d\xi$

$+ \frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{f(z_0)}{\xi - z} d\xi$

$\downarrow$ as $z \to z_0$

$\frac{1}{2\pi i} \int_{\Gamma - \Gamma_\varepsilon} \frac{f(\xi)}{\xi - z_0} d\xi$

$\downarrow$ as $\varepsilon \downarrow 0$

P.V.

$\forall \xi \in \Gamma_\varepsilon, \quad \frac{|\xi - z_0|}{|\xi - z|} \le C < \infty$ by non tangentiallity.

So $\int \frac{|f(\xi) - f(z_0)|}{|\xi - z|} = \underbrace{\left| \frac{f(\xi) - f(z_0)}{\xi - z_0} \right|}_{\text{bounded by } \max|f'| \text{ on } \Gamma_\varepsilon} \underbrace{\left| \frac{\xi - z_0}{\xi - z} \right|}_{\text{bounded for all } z \text{ by } C}$

So the second term is bounded as $z \to z_0$. As $\varepsilon \downarrow 0$ it will vanish.

Now we have $\frac{1}{2\pi i} \int_{\Gamma_\varepsilon} \frac{f(z_0)}{\xi - z} d\xi$. This is analytic by Morera.

So $= \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(z_0)}{\xi - z} d\xi = \frac{f(z_0)}{2\pi} \cdot$ aperture of C_ε. As $\varepsilon \downarrow 0$ aperture $\to \pi = 180°$ since Γ is C^1.

when $z \to z_0$

because we've moved the contour.

So as $\varepsilon \downarrow 0$, limit $= \frac{f(z_0)}{2}$. $\square$

Slightly different way: Extend f in C^1 way to a neighborhood of z_0 in $\mathbb{C}$.

$$\int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

$$= \int_{\Gamma \setminus \Gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \int_{\Gamma_\varepsilon} \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + \int_{C_\varepsilon} \frac{f(z)}{\zeta - z}\, d\zeta$$

$$\Big\downarrow z \to z_0 \qquad\qquad \Big\downarrow z \to z_0 \qquad\qquad \Big\downarrow z \to z_0$$

$$\underbrace{\int_{\Gamma \setminus \Gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z_0}\, d\zeta}_{I_1} + \underbrace{\int_{\Gamma_\varepsilon} \frac{f(\zeta) - f(z_0)}{\zeta - z_0}\, d\zeta}_{I_2} + \underbrace{\int_{C_\varepsilon} \frac{f(z_0)}{\zeta - z_0}\, d\zeta}_{I_3}$$

$$|I_2| \le C\varepsilon \qquad\qquad I_3 = i \cdot \text{Aperture of } C_\varepsilon$$

$$\Big\downarrow \qquad\qquad \Big\downarrow \varepsilon \to 0 \qquad\qquad \Big\downarrow \varepsilon \to 0$$

$$\text{p.v.} \int \frac{f(\zeta)}{\zeta - z_0}\, d\zeta \qquad\qquad 0 \qquad\qquad \pi i$$

Remark: 1 All limits as $z \to z_0$ justified by Dominated convergence.

2 Since $\lim_{\varepsilon \to 0} I_2$, $\lim_{\varepsilon \to 0} I_3$ exist, then $\lim_{\varepsilon \to} I_1$ exists so pv$\int$ exists for C' functions

3. I only need here $z \to z_0$, do not need non-tangentional approach