Jeff Miller

Recall from last time

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \xrightarrow[\substack{z\to z_0 \\ \text{nontangentially}}]{} \frac{1}{2\pi i}\,PV\int_\Gamma \frac{f(\zeta)}{\zeta - z_0}\,d\zeta + \frac{1}{2}f(z_0). \qquad f\in C^\alpha(\Gamma),\ \Gamma\in C^1.$$

In fact, this holds without assuming a nontangential approach:

1. Extend f to a C^1 f_n in a neighborhood of z_0.  z_0
2. $$\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \int_{\Gamma\setminus\Gamma_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \int_{\Gamma_\varepsilon} \frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta + \int_{\Gamma_\varepsilon} \frac{f(z)}{\zeta - z}\,d\zeta$$

   Take limit as $z\to z_0$.

   can replace by C_ε

Rk: Not essential to assume Ω is bounded, but must assume properties of f to guarantee convergence of the integral at ∞, e.g. if we assume f compactly supported, the given proof holds.

We have

$$P_+f(z) = \underbrace{\frac{1}{2\pi i}\,PV\int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2}f(z)}_{} \qquad z\in\mathbb{T},\ f\in C^\alpha(\mathbb{T})$$

Fourier representation: $\displaystyle\sum_{k\in\mathbb{Z}} \frac{1}{2}\,\mathrm{sign}(k)\,\hat{f}(k)\,z^k$ To see this, check it on z^n.

(where $\mathrm{sign}(0) = 1$)

Hilbert transform

$$(\mathcal{H}f)(z) = \frac{1}{\pi} PV \int_{\mathbb{T}} \frac{f(\zeta)}{z-\zeta} d\zeta$$

$$(\mathcal{H}f)(z) = -i \sum_{k\in\mathbb{Z}} \hat{f}(z)\, \mathrm{sign}(k)\, z^k$$

$f + i\mathcal{H}f$ - boundary values of H^2

$\shortparallel$

$2P_+f$

Define the operator $(Tf)(z) = \sum_{k\neq 0} -i\, \mathrm{sign}(k)\, \hat{f}(z)\, z^k$

letting $z = e^{i\theta}$, $\zeta = e^{it}$,

$$(Tf)(e^{i\theta}) = \frac{1}{\pi} PV \int_0^{2\pi} \frac{f(e^{it})\, ie^{it}}{e^{i\theta} - e^{it}} dt + \frac{i}{2\pi} \int_0^{2\pi} f(e^{it})\, dt$$

$(d\zeta = ie^{it} dt)$

$$= \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\left[\frac{2ie^{it}}{e^{i\theta} - e^{it}} + i \right]}_{\text{(kernel)}} f(e^{it})\, dt$$

$\shortparallel$

$$\frac{i(e^{i\theta} + e^{it})}{e^{i\theta} - e^{it}} = i\, \frac{e^{i(\theta-t)/2} + e^{i(t-\theta)/2}}{e^{i(\theta-t)/2} - e^{i(t-\theta)/2}}$$

$$= i\, \frac{2\cos(\theta-t)/2}{2i\sin(\theta-t)/2} = \cot\left(\frac{\theta-t}{2}\right)$$

$$\Rightarrow \quad (Tf)(e^{i\theta}) = \frac{1}{2\pi} PV \int_0^{2\pi} f(e^{it}) \cot\left(\frac{\theta - t}{2}\right) dt$$

This T is called the <u>Hilbert transform</u>.

(Note: $\mathcal{H}$ above is sometimes also called the Hilbert transform.)

Rk: If you apply the Sokhotsky formula on $\mathbb{R}$ you get

$$f(s) = \frac{1}{\pi} PV \int \frac{f(t)}{s-t}\, dt.$$

<u>Calderon - Zygmund Operators</u> (CZO)

Operator on $\mathbb{R}^n$ with kernel $K(s,t)$ satisfying:

1. $$|K(s,t)| \leq \frac{C}{|s-t|^n}$$

2. $\exists\, \alpha > 0$ s.t. $$\left.\begin{array}{l} |K(s,t) - K(s',t)| \\ |K(t,s) - K(t,s')| \end{array}\right\} \leq \frac{|s-s'|^\alpha}{|t-s|^{n+\alpha}}$$

$$|s - s'| < \frac{1}{2}|t-s|.$$

These arise naturally in PDE.

Also called "singular integral operators," because $\int f(t) K(s,t)\, dt$ is not always integrable. This is why we assumed f smooth above.

Facts about CZOs

① If CZO bdd in some L^{p_0} $p_0 \in (1,\infty)$,
then it is bdd in all L^p $p \in (1,\infty)$.

② If CZO is bdd, then PV exists a.e. for all $f \in L^p$ $(1<p<\infty)$

P_+ is bdd in L^2 ($\Rightarrow$ same for $\mathcal{H}$ and T)

$\Rightarrow$ They are bdd in all L^p $(1<p<\infty)$.

But none are bdd in L^∞ (take step fn to see this, using properties of cot.)

$\Rightarrow$ None are bdd in L^1
by duality.

Rk: $T^* = -T$ $\qquad I + i\mathcal{H} = 2P_+$.