Duals of H^p B.M.O.

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Thm for $1 < p < \infty$, $(H^p)^* \simeq H^q$, $\frac{1}{p} + \frac{1}{q} = 1$:

For any $\varphi \in (H^p)^*$ i.e. any linear bounded functional there exists a unique $g \in H^q$ such that $\varphi(f) = \int_{\mathbb{T}} f \bar{g} \frac{|dz|}{2\pi}$

and $\|\varphi\| \le \|g\|_q \le C_p \|\varphi\|$ not an isometry, but is a homeomorphism

by Hölder's inequality

$\varphi \in (H^p)^*$ we can extend φ by Hahn-Banach to a functional on L^p

$\Rightarrow$ exists $\tilde{g} \in L^q$, $\|\tilde{g}\|_q = \|\varphi\|$ such that $\int f \bar{\tilde{g}} \frac{|dz|}{2\pi} = \varphi(f)$ for all $f \in H^p$.

$\tilde{g}$ is definitely not unique because we can add any exponent and it will cancel out.

$g = P_+ \tilde{g}$ then $\|g\|_q \le C_p \|\tilde{g}\|_q$ and $\int f \bar{g} \frac{|dz|}{2\pi} = \varphi(f)$ for all $f \in H^p$

so we get equivalent norms, nothing interesting.

$(H^1)^* = \bar{X}$ where $X = P_+ L^\infty = \{ f : f = P_+ g : g \in L^\infty \}$ all P_+ projections of all bounded functionals

$\|f\| = \inf \|g\|_\infty$

Bounded mean oscillation

Def $f \in L^1_{loc}$, then $f \in BMO$ if for any interval I, there exists a universal constant C, and there is an a_I such that $⨍_I |f - a_I| \le C$

$\leftarrow \frac{1}{|I|} \int_I |f - a_I|$, average value on interval

The best constant C is called the BMO norm, denoted $\|f\|_\#$

(Here we are in one dimension, but in more dimensions, I is a cube instead of an interval.)

$f_I = ⨍_I f$ then

Claim $⨍_I |f - f_I| < 2 \|f\|_\#$

Remark $\sup_I ⨍ |f - f_I| \ge \|f\|_\#$

Proof of Claim $⨍_I |f - f_I| \le ⨍_I |f - a_I| + ⨍_I |\overbrace{f_I - a_I}^{\text{constant}}| = ⨍_I |f - a_I| + \left| ⨍ (f - a_I) \right| \le 2 \int |f - a_I|$

average of f over the interval is f_I so we can replace f_I with f

$\le C \le \|f\|_\#$.

moral of story:

$\|f\|_* = \sup_I ⨍_I |f - f_I|$ is an equivalent norm; very often in the literature, this is used as the definition of norm in BMO.

$$\|f\|_\# \le \|f\|_* \le 2\|f\|_\#$$

Examples

1. $L^\infty \subset BMO$
2. $f(x) = \ln|x| \Rightarrow f(x) \in BMO$ (Exercise: check this.) it is enough to check on $[0,R]$ (why?) then compute on those intervals
3. $f(x) = \begin{cases} \ln|x| & x > 0 \\ -\ln|x| & x < 0 \end{cases}$ $\Rightarrow f \notin BMO$ (look on symmetric interval, see difference)
   (or 0, $x < 0$) so the space is not ideal

Thm (John - Nirenberg)

There exist c, C (depending only on dimension) such that for all $f \in BMO$, for all I,

$$\frac{|\{x \in I : |f(x) - f_I| > t\}|}{|I|} \le C \exp\left(\frac{-ct}{\|f\|_*}\right).$$

i.e. it decreases exponentially as t grows, i.e. behaves logarithmically

Cor If $f \in BMO$, then $\left(⨍_I |f - f_I|^p\right)^{1/p} \le C_p \|f\|_*$.

Remark $⨍_I |f - f_I| \le \left(⨍_I |f - f_I|^p\right)^{1/p} \underbrace{\left(⨍_I 1^q\right)^{1/q}}_{=1}$ the reverse inequality is trivial.

Proof of Corollary

$F \ge 0$, $\int F^p \, d\mu = \int_0^\infty \mu\{x : F^p(x) > s\}\, ds$ change of variables $t^p = s$

$$= \int_0^\infty \mu\{x : F(x) > t\}\, p t^{p-1}\, dt$$

$$⨍_I |f - f_I|^p = \int_0^\infty \frac{|\{x : |f(x) - f_I| > t\}|}{|I|} p t^{p-1}\, dt \le \int C_p \exp\left(\frac{-ct}{\|f\|_*}\right) t^{p-1}\, dt$$

← lebesgue measure

introduce variable $\frac{ct}{\|f\|_*} = s$, $t = \frac{\|f\|_*}{c} s$

$$= \int_0^\infty C_p e^{-s} \frac{\|f\|_*^{p-1}}{c^{p-1}} s^{p-1} \overbrace{\frac{\|f\|_*}{c}\, ds}^{dt} = C_p \frac{\|f\|_*^p}{c^p} \underbrace{\int_0^\infty e^{-s} s^{p-1}\, ds}_{\text{just some number}}$$

So if your distribution function decays exponentially, it does not matter what norm we compute.

<u>Lemma</u> (Calderon - Zygmund decomposition)

Let $F \geq 0$ on I, $A > 0$ such that $\fint_I F = F_I < A$.

Then there exist a collection of disjoint subintervals $I_K \subset I$, $I_j \cap I_K = \phi$,

such that on any interval, $A \leq \fint_{I_K} F \leq 2A$ and $F \leq A$ a.e. on $I \setminus \bigcup_K I_K$.

Divide I in half. If your average on the interval is above A, stop; that is one of your I_K's.

If your average is less than A, you cannot jump more than $2A$.

Continue to divide it up and get the intervals where the average is small.