**Lemma.** $F \geq 0$ on $I$, $A > 0$.

$\fint F < A$

$\Rightarrow \exists\, I_k \subset I,\ I_k \cap I_j = \phi\ \ k \neq j$ s.t. $\begin{cases} A \leq \fint_{I_k} F < 2A \\ F \leq A \text{ on } I \setminus \cup I_k. \end{cases}$

**Remark.** If $A > 0$, $\fint_I F \leq 1$

then $\sum |I_k| \leq \frac{1}{A} |I|$.

pf. $\underset{\overset{\wedge|}{|I|}}{\int_I F} \geq \int_{\cup I_k} F = \sum_k (\fint_{I_k} F)\,|I_k| \geq A \sum |I_k|$. ⊞

pf of John-Nirenberg.

(WLOG $\|f\|_* = 1$)

$F := |f - f_I|$. $A = \frac{3}{2}$. $\fint_I F \leq 1$.

- $I_k^1$, $\frac{3}{2} < \fint_{I_k^1} |f - f_I| \leq 3$, $\sum |I_k^1| \leq \frac{2}{3} |I|$.

(↑ the first generation of C-Z decomposition)

$I \setminus \cup I_k^1 \quad F \leq \frac{3}{2}$.

- $\forall I_k^1 \quad F = |f - f_{I_k}| \quad A = \frac{3}{2}$

$I_j^2 \Rightarrow \sum |I_j^2| \leq (\frac{2}{3})^2 |I|$. $\quad \fint_{I_j^2} |f - f_I| \leq \overbrace{\fint_{I_j^2} |f - f_{I_k^1}|}^{\leq 3} + \overbrace{|f_{I_k^1} - f_I|}^{\leq 3} \leq 6$

because $|f_{I_k^1} - f_I| = |\fint_{I_k^1} f - f_I|$

$\leq \fint_{I_k^1} |f - f_I| \leq 3$

- On $I \setminus \cup I_j^2$, $|f(\cdot) - f_I| \leq |f - f_{I_k^1}| + |f_{I_k^1} - f_I| \leq \frac{3}{2} + 3 \leq 6$.

• • • •

By induction, we get $\{I_k^n\}$ such that $\sum |I_k^n| \leq (\frac{2}{3})^n I$ and $|f - f_I| \leq 3n$ on $I \setminus \cup I_k^n$ and $|f_{I_k^n} - f_I| \leq 3n$.

$t>0$ $\quad n \in \mathbb{N}$ s.t. $3n \le t \le 3(n+1)$. $\qquad c=\ln(\frac{3}{2})$ $\qquad t<3$

$\Rightarrow \{x : |f(x)-f_I(x)|>t\} \subset \bigcup I_k^n$.

$|\{x : |f(x)-f_I(x)|>t\}| \le (\frac{2}{3})^n |I| \le e^{-c(t-1)}|I|$

$0<t\le 3$ $\quad |\{x : |f(x)-f_I(x)|>t\}| \le |I| \le Ce^{-ct}$ $\qquad (C=e^{3c})$ $\qquad \boxplus$

Prop. $f \in BMO \Leftrightarrow \sup_{z\in\mathbb{D}} (|f-f(z)|^2(z))^{\frac{1}{2}} < \infty$ ; Garsia's norm.

Pf.
($\Leftarrow$)
$f(z)$ - harmonic extension of $f$, $\quad f(z)=\int \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} f(\zeta)\frac{|d\zeta|}{2\pi}$.

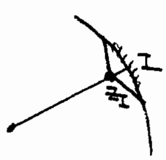
$\int_{\mathbb{T}} |f(\zeta)-f(z)|^2 \frac{1-|z|^2}{|1-\bar{z}\zeta|^2}\cdot\frac{|d\zeta|}{2\pi}$

Remark. 1. $\int |f(\zeta)-a|^2 \frac{1-|z|^2}{|1-\bar{z}\zeta|^2}\cdot\frac{|d\zeta|}{2\pi}$ $\quad$ ($z, f$ : fixed)

has its minimum at $a=f(z)$.

2. $\fint_I |f-a|^2$ minimum attained at $a=f_I$.

$I$ $\quad z_I$ $\qquad \frac{z_I}{|z_I|}$ : the center of $I$, $\quad 1-|z_I| = \frac{|I|}{2\pi}$.



$z=z_I$.

$\frac{1-|z|^2}{|1-\bar{z}\zeta|^2} \ge \frac{C}{|I|}$ $\quad \zeta \in I$

$\Rightarrow \int \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} |f-f(z)|^2 \frac{|d\zeta|}{2\pi} \le K$

$\Rightarrow \fint_I |f(\zeta)-f(z)|^2 \frac{|d\zeta|}{2\pi} \le \frac{1}{C}K$

$\qquad \vee|$

$\fint_I |f-f_I|^2$

$f \in BMO \Rightarrow \|f\|_G < \infty$ , i.e. $\|f\|_G \lesssim \|f\|_*$ .

$z$, $I_z$.

$$\int_{\mathbb{T}} \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} |f(\zeta)-f_{I_z}|^2 \frac{|d\zeta|}{2\pi} \lesssim C\|f\|_*^2 .$$

$I_z = I^0$ $\qquad$ $I^k = 2^k I_0$ $\quad$ (extend the interval "×2", keeping the center point,
For $\mathbb{T}$, it has finite intervals
" $\mathbb{C}_+$ " " infinite " ".)

$|f_{I^{k+1}} - f_{I^k}| \lesssim C\|f\|_*$ ,

$|f_{I^k} - f_{I^0}| \lesssim Cn\|f\|_*$ .

$$\left( \frac{1-|z|^2}{|1-\bar{z}\zeta|^2} \lesssim C\, 2^{-2k} \quad \text{if} \quad \zeta \in I_{k+1} \setminus I_k . \right)$$

$\fint_{I^k} |f - f_{I^0}| < C(n+1)\|f\|_*$

$\sum n 2^{-n} < \infty$ . $\quad \boxplus$