$\|f\|_* < \infty \Rightarrow \|f\|_G \leq C\|f\|_*$.

WLOG $\|f\|_* = 1$.

1. $\|f\|_* \leq 1 \Rightarrow |f_I - f_{2I}| \leq 2$

$$|f_I - f_{2I}| = \left|⨍_I f - f_{2I}\right| \leq \frac{1}{|I|}\int_I |f - f_{2I}| \leq \frac{2}{|2I|}\int_{2I} |f - f_{2I}| \leq 2\|f\|_* \leq 2.$$

So, $|f_I - f_{I^k}| \leq 2k$.

$$⨍_{I^k} |f_I - f_{I^k}| \leq ⨍_{I^k} |f - f_{I^k}| + |f_I - f_{I^k}| \leq 2k+1$$

$\Rightarrow$ $\left(⨍_{I^k} |f - f_I|^2\right)^{\frac{1}{2}} \leq C(2k+1)$

$⨍_{I^k} |f - f_I|^2 \leq C(2k+1)^2$

2. $J_k = I^k \setminus I^{k-1}$ $\quad k \geq 1$

$J_0 = I^0 = I$.

$$P_z(\xi) = \frac{1-|z|^2}{|1-\bar{z}\xi|^2} \leq \frac{C}{|I|}(2^{-k})^2 \quad \text{for } \xi \in J_k.$$

$$\leq \frac{C}{|I^k|} 2^{-k}.$$

3. $$\int_{\mathbb{T}} P_z(\xi)\, |f(\xi) - f_I|^2 \frac{|d\xi|}{2\pi} = \sum_{k\geq 0} \int_{J_k} \cdots$$

$$\leq \sum \frac{C}{|I^k|} 2^{-k} \int_{I^k} |f(\xi) - f_I|^2 \frac{|d\xi|}{2\pi} \leq C \sum 2^{-k} C(k+1)^2 < \infty$$

- Why is the Garcia's norm is useful?

$\Rightarrow$ Garsia's norm is conformally invariant, i.e. if φ : Möbius transform, then $\|f \circ \varphi\|_G = \|f\|_G$.

$$\|f\|_G^2 \stackrel{\text{def}}{=} \sup_z |f - f(z)|^2(z) \stackrel{\text{Exercise.}}{=} \sup_z |f|^2(z) - |f(z)|^2$$

$\Rightarrow$ BMO is a conformally invariant space.

- $\|f\|_G < \infty \Leftrightarrow \forall \varphi$ - Möbius $\inf_a \|f \circ \varphi - a\|^2 \leq \|f\|_G^2$.

Lemma. If $g \in BMO$, $g(z)$ is a harmonic extension of $g_{\mathbb{T}}$,

then $|\nabla g(z)|^2 (1-|z|^2)\, \underbrace{dx\,dy}_{=dA(z)}$ is Carleson.

Pf. (Use Green's formula)

$$\Delta |g(z)|^2 = 2|\nabla g(z)|^2$$

↑ computation $\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_k}(g\bar{g}) = \cdots$

$$|g|^2(\lambda) - |g(\lambda)|^2 = \frac{1}{2\pi}\int_{\mathbb{D}} \Delta(|g(z)|^2) \ln\left|\frac{1-\bar{\lambda}z}{z-\lambda}\right| dA(z)$$

$$= \frac{1}{2\pi}\int_{\mathbb{D}} \cancel{2}|\nabla g(z)|^2 \tfrac{1}{\cancel{2}} \ln\left|\frac{1-\bar{\lambda}z}{z-\lambda}\right|^2 dA(z)$$

$$\geq \frac{1}{2\pi}\int_{\mathbb{D}} |\nabla g(z)|^2 \frac{(1-|\lambda|^2)(1-|z|^2)}{|1-\bar{\lambda}z|^2}\, dA(z) \cdots \rightarrow \text{Carleson measure condition.}$$

For more precise proof, we need to apply ↑ to $g_r(z) = g(rz)$, $r \to 1$. ⊞

Remark. "Lemma" is actually "iff".

Lemma. If $f \in H^2$, then $\|f\|_2^2 = |f(0)|^2 + \frac{2}{2\pi}\int_{\mathbb{D}} |f'(z)|^2 \ln\frac{1}{|z|}\, dA(z)$.

Pf. Apply Green's formula to $|f(z)|^2$. ⊞

~~Prop. Let g ∈ Harm(D) ...~~,

~~g(0) = 0~~,

~~If |∇g(z)|²~~

Prop. Let $g \in BMO$. $g(0) = 0$. (harm. ext.)

$Lf = \int_{\mathbb{T}} g f \frac{|dz|}{2\pi}$; a linear functional on H^1.

$\Rightarrow |Lf| \leq C\|g\|_* \|f\|_{H^1}$.

Pf. $\int_{\mathbb{T}} g f \frac{|dz|}{2\pi} = \frac{1}{2\pi}\int_{\mathbb{T}} \Delta(gf) \ln\frac{1}{|z|}\, dA(z) \overset{\Delta = 4\partial\bar{\partial}}{=} \frac{2}{\pi}\int_{\mathbb{D}} \partial g\, f' \ln\frac{1}{|z|}\, dA(z)$

$\begin{cases} f = f_1 f_2 & f_1, f_2 \in H^2. \\ \|f_1\|_2^2 = \|f_2\|_2^2 = \|f\|_1 \end{cases}$

$$f' = f_1 f_2' + f_2 f_1'$$

$$\int_{\mathbb{D}} |\bar{\partial} g| \cdot |f_1 f_2'| \ln\frac{1}{|z|}\, dA(z) \le \left(\int_{\mathbb{D}} |\nabla g|^2 |f_1|^2 \ln\frac{1}{|z|}\, dA(z)\right)^{\frac{1}{2}} \left(\int_{\mathbb{D}} |f_2'|^2 \ln\frac{1}{|z|}\, dA(z)\right)^{\frac{1}{2}}$$

$\le C\|f_1\|_2$ $\qquad\qquad$ $\le C\|f_2\|_2$

↑ because Carleson measure condition.

(Divide $\{|z| \le \frac{1}{2}\}$ and $\mathbb{D} \setminus \{|z| \le \frac{1}{2}\}$

...).