Recall: $g \in BMO$, $g(0)=0$

We defined $Lf = \int_{\mathbb{T}} g f \frac{|dz|}{2\pi}$ for $f \in H^1$.

$|Lf| \le C \|g\|_* \|f\|_{H^1}$.

$f = f_1 f_2$, $f_1, f_2 \in H^2$ and $\|f_1\|_2^2 = \|f\|_1 = \|f_2\|_2^2$.

$$\int_{\mathbb{D}} |\bar\partial g| |f_1 f_2'| \ln\frac{1}{|z|}\, dA(z) \le \left(\int_{\mathbb{D}} |\nabla g|^2 |f_1|^2 \ln\frac{1}{|z|}\, dA(z)\right)^{\frac12} \times \left(\int_{\mathbb{D}} |f_2'|^2 \ln\frac{1}{|z|}\, dA(z)\right)^{1/2}$$

$$\le \|f_2\|_2.$$

We know $|\nabla g|^2 (1-|z|)\, dA(z)$ is Carleson

$$\int_{\mathbb{D}} |f_1|^2 |\nabla g| \ln\frac{1}{|z|}\, dA(z) = \int_{\frac12 \le |z| < 1} \_\_\_\_ + \int_{|z|<\frac12}.$$

If $|z| \in [\frac12, 1) \Rightarrow \ln\frac{1}{|z|} \le C(1-|z|)$ $(C = e\ln 2)$.

Then $\int_{\frac12 \le |z| < 1} |f_1|^2 |\nabla g| \ln\frac{1}{|z|}\, dA(z) \le C \|g\|_* \|f_1\|_2^2 = C\|g\|_* \|f\|_1$.

On $|z| < \frac12$: then $\frac{1-|z|^2}{|1-\bar z \xi|} \le C$. (for $\xi \in \mathbb{T}$).

$\Rightarrow |f_1(z)| \le C \int_{\mathbb{T}} |f_1| \frac{|dz|}{2\pi} \le C\|f_1\|_2$ (L^1 norm bounded by L^2 norm)

$\Rightarrow |g(z)| \le C\|g\|_1 \le C\|g\|_*$.

So $\int_{|z|<\frac12} \cdots \le C \cdot \|f_1\|_2^2 \cdot \|g\|_* \underbrace{\int_{|z|<\frac12} \ln\frac{1}{|z|}\, dA(z)}_{\text{finite constant.}}$

We've estimated one factor, bound the second factor similarly.

Upshot: BMO functions give bounded linear functionals.

Proposition: Let $g_-(z) = \sum_{k \ge 0} c_{-k} z^{-k}$, $g_- \in L^2$. s.t. $Lf = \int_{\mathbb{T}} g_- f \frac{|dz|}{2\pi}$ is ~~[illegible]~~ in $(H^1)^*$, i.e. $\|Lf\| \le C\|f\|_{H^1}$ $\forall f \in H^1$. Then $g_- \in BMO$ and $\|g_-\|_* + |\hat g_-(0)| < C\|L\|$.

Pf: $|\hat g_-(0)| \le \|L\|$ trivial — test the functional on $f = 1$.

$L \in (H^1)^*$, so we can extend L to a functional on L^1.

$\Rightarrow \exists\, g \in L^\infty$, $\|g\|_\infty = \|L\|$ s.t. $Lf = \int_{\mathbb{T}} f g \frac{|dz|}{2\pi}$ $\forall f \in H^1$.

$\Rightarrow g = g_- + g_+$, $g_+ \in H^2$.

Pf continued:

Recall: Uchiyama's lemma:

If u is bounded, subharmonic, $\Delta u \geq 0 \Rightarrow \Delta u \ln\frac{1}{|z|}\, dA(z)$ is Carleson.

with all constants estimated by $C\|u\|_\infty$.

Apply Uchiyama's lemma to $u = |g(z)|^2$.

$\Delta |g|^2 = 2|\nabla g|^2 \geq 0.$

For any f, $|\nabla f|^2 = 2(|\partial f|^2 + |\bar\partial f|^2)$.

$\Rightarrow$ $|\nabla g|^2 \ln\frac{1}{|z|}$ is carleson $\Rightarrow |\bar\partial g|^2 \ln\frac{1}{|z|}\, dA(z)$ is carleson.

$\bar\partial g = \bar\partial g_-$, so $|\bar\partial g_-|^2 \ln\frac{1}{|z|}\, dA(z)$ is Carleson.

$$|g_-|^2(0) - |g_-(0)|^2 = \frac{1}{2\pi}\int_{\mathbb{D}} \Delta|g_-|^2 \ln\frac{1}{|z|}\, dA(z) \qquad \text{(Green's formula)}$$

$$= \frac{2}{\pi}\int_{\mathbb{D}} |\bar\partial g_-|^2 \ln\frac{1}{|z|}\, dA(z)$$

$$\leq C\|g\|_\infty^2 \quad \text{since } |\bar\partial g_-|^2 \ln\tfrac{1}{|z|} \text{ is Carleson}$$

$$= C\|L\|^2.$$

Consider $\tilde g = g \circ b_\lambda^{-1}$.

$\tilde g_+ = g_+ \circ b_\lambda^{-1}$ and $\tilde g_- = g_- \circ b_\lambda^{-1}$

$\tilde g_+ \in H^2$, $\tilde g_- \in H^2_- + \text{constants}$.

Applying the same reasoning to $\tilde g, \tilde g_-, \tilde g_+$, we get

$$|g_-|^2(\lambda) - |g_-(\lambda)|^2 = |\tilde g_-|(0) - |\tilde g_-(0)|^2 \leq C\|\tilde g\|_\infty^2 = C\|g\|_\infty^2.$$

Alternative Proof: Consider the Hankel operator $H_g = H_{g_-}$.

$k_\lambda = \dfrac{(1-|\lambda|^2)^{1/2}}{1-\bar\lambda z}$, $\|k_\lambda\|_{H^2} = 1$ (normalized reproducing kernel).

$$H_{g_-} k_\lambda = g_- k_\lambda - P_+(g_- k_\lambda) = g_- k_\lambda - T_{g_-} k_\lambda$$

$$= g_- k_\lambda - g_-(\lambda) k_\lambda.$$

But $H_{g_-} k_\lambda \perp T_{g_-} k_\lambda$.

So $$\|H_{g_-} k_\lambda\|_2^2 = \|g_- k_\lambda\|_2^2 - \|g_-(\lambda) k_\lambda\|^2$$

$$= \int |g_-(\xi)|^2 \frac{1-|\lambda|^2}{|1-\bar\lambda\xi|^2}\frac{|d\xi|}{2\pi} - |g(\lambda)|^2$$

$$= |g_-|^2(\lambda) - |g(\lambda)|^2 \leq \|g\|_\infty^2 \quad \text{(since it is bounded by the norm of the Hankel operator.}$$

Remark: Can repeat these proofs for functions on $\mathbb{R}$.