Corona theorem

Maximal ideals

Banach algebra: X - Banach space, complex valued, with addition & multiplication by scalars, has a norm,

and $f \cdot g$ - multiplication continuous in each variable (f, g) separately.

so for all f, $\|fg\| \le C_f \|g\|$

and for all g, $\|fg\| \le C_g \|f\|$.

By Uniform Boundedness Principle, we can conclude that

$\|fg\| \le C \|f\| \|g\|$. $(*)$

<u>Remark 1</u> If X is a unital Banach Algebra (has a unit: $\exists e : \overset{fe}{\overset{\shortparallel}{ef}} = f$ for all $f \in X$)

then ~~there~~ exists an equivalent norm on X where

$$\|fg\|_0 \le \|f\|_0 \cdot \|g\|_0$$

<u>Pf</u> Introduce an operator M_f: $M_f g = fg$.

and a norm $\|f\|_0 = \|M_f\|$. Then the above inequality is trivial.

Why are the norms equivalent?

$\|M_f\| \le C \|f\|$ by $(*)$

and on the other hand, $\|M_f\| = \sup\{\|fg\| : g \in X, \|g\| \le 1\} \ge \|fe\| = \|f\| \underbrace{\|e\|}_{\neq 0}$

so we in fact have equivalent norms.

So we have $\|fg\| \le \|f\| \|g\|$ and $\|e\| = 1$.

<u>Remark 2</u> Arbitrary X can be embedded into a unital Banach algebra.

If your algebra has no unit, take an element from a different alphabet and call it the identity; then your new algebra is the original with all the products with the identity.

Gelfand developed this, also someone else much earlier who claims that the KGB stole his ideas.

X - commutative, unital Banach algebra, $fg = gf$.

<u>Example 1</u> H^∞ bounded functions in the unit disk.

2. $C^{(n)}[a,b]$

3. Analytic $\overset{\text{(Norbert)}}{\text{Wiener}}$ algebra $W_A = \{f : f = \sum_{k \ge 0} a_k z^k\}$, $\|f\| = \sum |a_k|$

converges uniformly in $\mathbb{D}$ and continuous up to the boundary

4. Non-continuous Wiener algebra $W = \{f : f = \sum_{k \in \mathbb{Z}} a_k z^k\}$ $\|f\| = \sum |a_k|$, $z \in \mathbb{T}$.

It is easy to check that these are Banach algebras with the usual multiplication of functions.

Ideal: $I \subset X$ such that $x, y \in I \Rightarrow \alpha x + \beta y \in I$ for all scalars $\alpha, \beta \in \mathbb{C}$
and for all $x \in X$, $xI \subset I$ ($xy \in I$ for all $y \in I$).

(assume commutative algebra, so no distinction between right and left ideals)

I = ideal, then X/I = algebra

I = closed ideal and X is closed Banach algebra (CBA) $\Rightarrow$ X/I is CBA.

Lemma If I is ideal and x is invertible, $x \in I$ $\Rightarrow I = X$.

Pf If $e \in I \Rightarrow x \in xI$ for all $x \in X$,

and if there exists x^{-1} then $e \in x^{-1} I$.

So any proper ideal does not contain an invertible element.

Maximal ideals

If I is maximal, then $I_1 \supsetneqq I \Rightarrow I_1 = X$.

Defn $\varphi \in X^*$ is called multiplicative if $\varphi(xy) = \varphi(x)\varphi(y)$.

If φ is a multiplicative linear functional (MLFnl) then $\ker \varphi$ is a maximal ideal.

For all maximal ideals I, there exists a unique φ (MLFnl) such that $I = \ker \varphi$.

(So there is a 1-to-1 correspondence between MLFnls and maximal ideals.)

1. I maximal $\Rightarrow I$ closed.

$I \subset \mathrm{cl}\, I \subset X$. I contains no invertible elements $\Rightarrow \{x : \|x - e\| < 1/2\} \Rightarrow x \notin I$

because all such x are invertible.
(write inverse as sum of geometric series)

$\Rightarrow e \in \mathrm{Int}\, X \setminus I \Rightarrow e \notin \mathrm{clos}\, I$

$\Rightarrow \mathrm{cl}\, I \neq X \Rightarrow I = \mathrm{cl}\, I$ since I is maximal.

X/I is a CBA and Field $\simeq \mathbb{C}$.

Thm (Wiener invertibility theorem)

$f \in W_A$, $f(z) \neq 0$ for all $z \in \mathbb{D} \Rightarrow 1/f \in W_A$.

1. $\mathfrak{M}_{W_A} = \overline{\mathbb{D}}$ (maximal wiener algebra space is $\overline{\mathbb{D}}$):

$\lambda \in \overline{\mathbb{D}}$, $\varphi_\lambda(f) = f(\lambda)$ is linear, fix λ, then for any f is linear.

so all functionals are just point evaluations.

$\varphi(z) = \lambda$, then it is easy to show that if φ is a MLFnl $\Rightarrow \|\varphi\| = 1$.

$\Rightarrow \|\lambda\| \le 1$. Since φ is a MLFnl, $\varphi(p) = p(\lambda)$ for all polynomials p.

Since polynomials are dense, $\Rightarrow \varphi(f) = f(\lambda)$ for all $f \in W_A$.

If f is not invertible $\Rightarrow$ ideal generated by f $I_f = \{ fg : g \in W_A \} \not\ni 1$ proper ideal

$\Rightarrow f$ belongs to some maximal ideal of W_A

$\Rightarrow \exists\, \lambda : f(\lambda) = 0$.

$\mathcal{M}_{C^{(n)}[a,b]} = [a,b]$ ← maximal ideal space

$\mathcal{M}_W = \mathbb{T}$

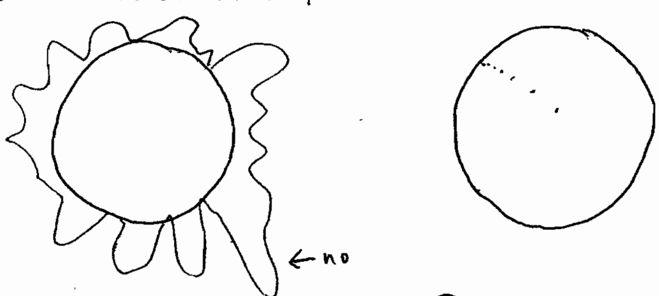
what can be said about $\mathcal{M}_{H^\infty}$? $\mathbb{D} \subset \mathcal{M}_{H^\infty}$

Kakutani asked: Is $\mathbb{D}$ dense in $\mathcal{M}_{H^\infty}$?

Carleson answered: Yes.

To visualize how $\mathcal{M}_{H^\infty}$ looks: is it just on the edge of the disk, or are there pieces that stick out like a sun corona?

Corona theorem says there is no corona.



What does it mean to say $\mathbb{D}$ is dense in $\mathcal{M}_{H^\infty}$? We will see.

We will get questions that are very simple to state and very difficult to answer.

If you replace $\mathbb{D}$ by a ball, even in $\mathbb{C}^2$, nobody knows the answer.