Last time introduced maximal ideals,
and discussed the Corona theorem.

<u>Topology of max. ideals space</u> ("Gelfand Topology")

Same as weak $*$ topology inherited from the dual space X^*.

To define a base, it is enough to define it at an arbitrary point, such as 0.

Base of top at 0 consists of sets of form

Let $B_0 = \{ f \in X^* \mid |f(x_k)| < \varepsilon \ \forall k, \ x_1, \dots, x_n \in X, \ \varepsilon > 0 \}$

parametrized by $\varepsilon, x_1, x_2 \dots x_n$

Let $B_{f_0} = \{ f \in X^* \mid |(f - f_0)(x_k)| < \varepsilon \ \forall k, \ x_1, x_2, \dots, x_n \in X, \ \varepsilon > 0 \}$

$\mathcal{M}_X$ Base of top at f_0 consists of sets of form $B_{f_0, \varepsilon, x_1, \dots, x_n}$

Closed unit ball of X^* is compact in $w*$ topology.

$\mathcal{M}_X$ is $w*$ closed subset of Ball of X^*

$\mathcal{M}_X \subset$ Ball

$f(xy) - f(x)f(y) = 0 \quad \forall x, y \in X$

<u>Rk!</u> Unit sphere $\{ f \in X^* \mid \|f\| = 1 \}$ <u>not</u> compact in $w*$ topology if $\dim X = \infty$.

$\mathbb{D}$ not dense in $\mathcal{M}_{H^\infty}$

$\Leftrightarrow \exists\, m_0 \in \mathcal{M}_{H^\infty}$ s.t. $m_0 \in \operatorname{Int}(\mathcal{M} \setminus \mathbb{D})$

$\Leftrightarrow \exists\, m_0 \in \mathcal{M}\ \exists\, \varphi_1, \dots, \varphi_n \in H^\infty,\ \exists\, \delta > 0$ s.t.

$$\{ m \in \mathcal{M} \mid |(m - m_0)(\varphi_k)| < \varepsilon \ \ \forall k = 1, \dots, n \} \cap \mathbb{D} = \emptyset$$

$\Leftrightarrow \exists\, m_0 \in \mathcal{M}\ \exists\, \varphi_1, \dots, \varphi_n \in \mathcal{M},\ \exists\, \delta > 0$ s.t.

$$\max_k |m_z(\varphi_k) - m_0(\varphi_k)| \geq \delta, \qquad m_z(\varphi) = \varphi(z).$$

$\Leftrightarrow \exists\, m_0 \in \mathcal{M}\ \exists\, f_1, \dots, f_n \in H^\infty$ s.t.

$$\max_k |f_k(z)| \geq \delta \ \ \forall z \in \mathbb{D} \quad \text{and} \quad \underbrace{m_0(f_k) = 0 \ \ \forall k.}$$

$$\Downarrow$$

$f_1, \dots, f_n \in I \subsetneq H^\infty$ some ideal I.

So $\nexists\, g_1, \dots, g_n \in H^\infty$ s.t. $\displaystyle\sum_{k=1}^{n} f_k g_k \equiv 1$.

This means that $\mathbb{D}$ is not dense in $\mathcal{M}_{H^\infty}$ if

$\exists\, f_1, \dots, f_n \in H^\infty,\ \delta > 0$ s.t. $\max_k |f_k(z)| \geq \delta > 0 \ \ \forall z \in \mathbb{D}$

and $\sum f_k g_k \not\equiv 1 \ \ \forall g_1, \dots, g_n \in H^\infty$.

Theorem (Carleson Corona Theorem)

Given $f_1, \dots, f_n \in H^\infty$ s.t. $\sum_{k=1}^{n} |f_k(z)|^2 \geq \delta^2 > 0 \quad \forall z \in \mathbb{D}$

$\exists\, g_1, \dots, g_n \in H^\infty$ s.t. $\sum |g_k(z)|^2 \leq C(\delta)^2$

and $\sum f_k g_k \equiv 1$.

Note, $C(\delta)$ does not depend on n.

Rk: Best known constant: $C(\delta) \leq C\,\delta^{-2} \ln 1/\delta$,

and it is known that $\delta^{-2} \ln \ln 1/\delta \geq C(\delta)$.

More generally, we have the

Matrix "Corona" problem:

Given $F \in H^\infty_{m\times n}$ 

("tall" matrix)

s.t. $F^* F \geq \delta^2 I_{n\times n}$, does there exist

$G \in H^\infty_{n\times m}$ s.t. $GF \equiv I$.

Rk: Carleson thm covers the case 

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Let $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ $\qquad \varphi(z) = \dfrac{1}{|f(z)|^2} f(z)^*$ ⤳ Hermitian

Then $\varphi(z) \; f(z) = 1$,

$$\bar\partial \varphi = \frac{(f')^*}{|f(z)|^2} - \frac{(f')^* f}{|f|^4} f^*$$

Define $\qquad \Phi = \varphi^T \bar\partial \varphi \qquad (n \times n)$

Suppose $\qquad \bar\partial \Psi = \Phi \qquad$ for some Ψ.

Then $\qquad g = \varphi + f^T(\Psi^T - \Psi)$

$\qquad\qquad g \in H^\infty$ and $gf \equiv 1$.

For, $\quad gf = \underbrace{\varphi f}_{\equiv 1} + \underbrace{\cancel{f^T(\Psi^T - \Psi) f}}_{0}$

and $g \in H^\infty$ because:

$$\bar\partial g = \bar\partial\varphi + f^T\left((\bar\partial \varphi^T)\varphi - \varphi^T \bar\partial \varphi \right)$$

$$= \underbrace{\bar\partial\varphi + (\bar\partial \cancel{(\underbrace{\varphi f}_{=1})^T}) \varphi}_{0} - \underbrace{f^T\varphi^T}_{=1} \bar\partial\varphi \quad = 0.$$