$$\bar\partial \psi = \varphi = \varphi^T \bar\partial \varphi$$

$$g = \varphi + f^T(\psi - \psi^T)$$

$$gf \equiv 1.$$

OBSERVATIONS.

1. $g^r(z) f(rz) \equiv 1$

$|g^r(z)| \leq C$ — usual argument of the normal family.

$g^{r}_{k} \to g$ uniformly compact subset of $\mathbb{D}$

$gf \equiv 1.$

WLOG, we can assume f_k analytic in $\operatorname{Clos} \mathbb{D}$ $\quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$

2. It suffices to show that $\psi \in L^\infty(\mathbb{T})$.

3. Operator norm = max singular value.

Frobenius norm $\quad \|A\|_{\mathcal{F}}^2 = \operatorname{tr} A^* A = \sum |a_{jk}|^2 = \sum |s_k|^2.$

Hilbert - Schmidt norm.

$\Rightarrow \|A_{op}\|_{op} \leq \|A\|_{\mathcal{F}}$

$\Rightarrow$ So, we'll show $\|\cdot\|_{\mathcal{F}}$ is bounded.

4. $\bar\partial F = \Phi$
$\Rightarrow F(z) = \frac{1}{2\pi i} \iint_{\mathbb{D}} \Phi(\xi) \frac{1}{\xi - z} d\xi \wedge d\bar\xi$

We want to find $\underbrace{\frac{\partial G}{\partial \bar\xi} = \delta_z}$; Fundamental sol

$\Delta_\xi \frac{-1}{2\pi} \ln|\xi - z| = \delta_z. \qquad \Delta = \partial \bar\partial$

$= \bar\partial \left(\partial \frac{-1}{2\pi} \ln|\xi - z| \right)$

5. Remark about vector-valued

H^p, H^p_n, $\mathbb{C}^n$,

$$f=\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \qquad f_k \in H^p$$

$$\|f\| = \sup_{0<r<1} \left(\int \|f(re^{it})\|^p_{\mathbb{C}^n} \frac{dt}{2\pi} \right)^{\frac{1}{p}} = \left(\int_{\mathbb{T}} \|f(\xi)\|^p_{\mathbb{C}^n} \frac{|d\xi|}{2\pi} \right)^{\frac{1}{p}}.$$

• $H^p_{n\times n} \cong H^p_{n^2}$, (by considering $\|\cdot\|_{\mathcal{Z}}$)

$f \in H^p_n$.

$$\int \ln \|f(\xi)\|_{\mathbb{C}^n} \frac{|d\xi|}{2\pi} > -\infty$$

$$F = \exp\left(\int_{\mathbb{T}} \frac{\xi+z}{\xi-z} \ln \|f(\xi)\| \frac{d\xi}{2\pi} \right)$$

$F \in H^p$, outer. $|F(z)| = \|f(z)\|$ a.e. on $\mathbb{T}$.

$\|f(z)\|_{\mathbb{C}^n} \le |F(z)| \quad \forall z \in \mathbb{D}$.

a) If μ is Carleson measure $\forall f \in H^p_n$,

$$\int_{\mathbb{D}} \|f(z)\|^p_{\mathbb{C}^n} d\mu(z) \le C \|f\|^p_{H^p_n}. \qquad 1 \le p < \infty.$$

b) $f \in H^1_n$. $\Rightarrow$ $F = f_1 f_2$ $\quad f_1 \in H^2$, $f_2 \in H^2_n$

$$\|f_1\|_2^2 = \|f_2\|_2^2 = \|f\|_{H^1_n}.$$

$f_1 = \sqrt{F}$ (we can do this, since $F \neq 0$ for any z).

$\underline{\underline{L}}$ Let $|\Phi(z)|^2 \ln \frac{1}{|z|} dA(z)$

Carleson, with constant C_1

$|\partial \varphi(z)|^2 \ln \frac{1}{|z|} dA(z)$

Carleson, with constant C_2

Then, $\bar{\partial} \psi = \varphi$ has a bounded solution $\|\psi\|_{L^\infty(\mathbb{T})} \le C(\sqrt{C_1} + \sqrt{C_2})$.

Remark. If in addition, $|\Phi(z)|^2 \leq \frac{C_3}{(1-|z|)^2}$

$|\partial\Phi(z)|^2$

$$\bar\partial \varphi = \frac{(f')^*}{\|f\|_{\mathbb{C}^n}} - \frac{(f')^* f f^*}{\|f\|^4_{\mathbb{C}^n}}$$

$\Phi = \varphi^T \bar\partial\varphi$

$\|\Phi(z)\|_{\mathcal{F}} = \|\varphi(z)\|_{\mathbb{C}^n} \|\bar\partial\varphi(z)\|_{\mathbb{C}^n} \leq \|\varphi\|_\infty \|\bar\partial\varphi(z)\|_{\mathbb{C}^n}. \leq C\|f'(z)\|_{\mathbb{C}^n}$

$\Rightarrow \left(\sum_{k,j} |\varphi_k(z)\bar\partial\varphi_j(z)|^2\right)^{\frac{1}{2}}$

$u(z) = \cancel{\|\varphi(z)\|^2_{\mathbb{C}^n}} \quad \|f(z)\|_{\mathbb{C}^n}$

$\Delta u(z) = \cancel{2\|\nabla\varphi\|^2_{\mathbb{C}} = 2\sum\left(\left|\frac{\partial\varphi_k}{\partial x_k}\right|^2 + \left|\frac{\partial\varphi_k}{\partial y_k}\right|^2\right)} = 4\|f'(z)\|_{\mathbb{C}^n}$

$\|\bar\partial\varphi(z)\|^2 \leq C\|\nabla\varphi(z)\|^2_{\mathbb{C}^n}$

By Uchiyama's lemma, $\|\nabla\varphi(z)\|^2_{\mathbb{C}^n} \ln\frac{1}{|z|} dA(z)$

$\Rightarrow |\Phi(z)|^2 \ln\frac{1}{|z|} dA(z)$ Carleson.

$\|\partial\Phi(z)\|_{\mathcal{F}} \leq C\|f'(z)\|^2_{\mathbb{C}^n}$. $\qquad u(z) = |f(z)|^2$. $\qquad$ Similarly

$\varphi(z) = \frac{f(z)^*}{|f(z)|^2}$.

$\bar\partial\varphi = \frac{(f')^*}{|f|^2} - \frac{(f')^* f}{|f|^4} f^* \qquad |f|^2 = f^* f$

$\qquad |f|^4 = (f^* f)^2$.

So $\|\partial\varphi\|_{\mathcal{F}} \ln\frac{1}{|z|} dA(z)$ - Carleson