<u>Lemma</u>: Φ s.t. $|\Phi|^2 \ln\frac{1}{|z|}\, dA$ is Carleson with constant C_1
and $|\partial\Phi| \ln\frac{1}{|z|}\, dA$ is Carleson with constant C_2
Then $\bar\partial\Psi = \Phi$ has a solution Ψ s.t. $\|\Psi\|_{L^\infty(\mathbb{T})} \le C(\sqrt{C_1} + C_2)$.

Explanation (correction from last time).

$$\Phi = \varphi^T \bar\partial \varphi \quad \text{and} \quad \|\Phi(z)\|_{\mathcal{F}} \le C\|f'(z)\|_{\mathbb{C}^n}.$$

Solving $\bar\partial\Psi = \Phi$ is not a problem — hard part is the bound.
So let Ψ_0 be some solution (the solution is not unique; we can add a holomorphic fn).

$$\operatorname{Inf}\left\{\|\Psi_0 + g\|_{L^\infty(\mathbb{T}, M_{n\times n})} \;\middle|\; g \in H^\infty_{n\times n}\right\} = \sup\left\{\int_{\mathbb{T}} \operatorname{tr}(\Psi_0 h)\frac{|dz|}{2\pi} \;\middle|\; h \in zH^1_{n\times n}\right\}. \quad (\star)$$

$(L^1)^* = L^\infty$ and $(L^1_{\mathbb{C}^n})^* = L^\infty_{\mathbb{C}^n}$.

In general, $(L^1_X)^* = L^\infty_{X^*}$ if X has the Radon-Nikodim property.

If $L(h) = \int_{\mathbb{T}} \operatorname{tr}(\Psi_0 h)\frac{|dz|}{2\pi}$, extend L to L^1 by Hahn-Banach.
So $\exists\, \Psi \in L^\infty$ s.t. $L(h) = \int \operatorname{tr}(\Psi h)\frac{|dz|}{2\pi}$ $\forall h \in zH^1_{n\times n}$ and $\|\Psi\|_{L^\infty(\mathbb{T})} = \|L\|$.
But $L(h) = \int \operatorname{tr}(\Psi_0 h)\frac{|dz|}{2\pi}$, so $\Psi - \Psi_0 \in H^\infty$, and we get the inequality needed for $(\star)$.

$$L(h) = \int \operatorname{tr}(\Psi h)\frac{|dz|}{2\pi} = \frac{2}{\pi}\int_{\mathbb{D}} \left[\partial\bar\partial \operatorname{tr}(\Psi h)\right] \ln\frac{1}{|z|}\, dA \quad \text{(Green's formula)}.$$

$$= \frac{2}{\pi}\int_{\mathbb{D}} \partial \operatorname{tr}(\Phi h) \ln\frac{1}{|z|}\, dA(z)$$

$$= \frac{2}{\pi}\int_{\mathbb{D}} \operatorname{tr}(\partial\Phi \cdot h) \ln\frac{1}{|z|}\, dA(z) + \frac{2}{\pi}\int_{\mathbb{D}} \operatorname{tr}(\Phi h') \ln\frac{1}{|z|}\, dA(z)$$

$$= \mathrm{I} + \mathrm{II}.$$

$$|\mathrm{I}| \le \frac{2}{\pi}\int \|\partial\Phi(z)\|_{\mathcal{F}} \cdot \|h(z)\|_{\mathcal{F}^*} \ln\frac{1}{|z|}\, dA(z) \leftarrow \text{Carleson measure}$$

$$\le C\cdot C_2 \|h\|_{H^1_{n\times n}}.$$

For $|\mathrm{II}|$: $h \in zH^1_{n\times n} \subset H^1$, so $h = h_1 h_2$ where $h_1 \in H^2$, $h_2 \in H^2_{n\times n}$.
and $\|h_1\|_2^2 = \|h_2\|_2^2 = \|h\|_1$.

So $$|\mathrm{II}| \le \frac{2}{\pi}\int_{\mathbb{D}} |\Phi|_{\mathcal{F}} \, |(h_1 h_2' + h_1' h_2)|_{\mathcal{F}} \ln\frac{1}{|z|}\, dA(z)$$

We have $|\mathrm{II}| \le \frac{2}{\pi}\int_{\mathbb{D}} |\Phi|_{\mathcal{F}} \, |h_1 h_2' + h_1' h_2|_{\mathcal{F}} \ln\frac{1}{|z|}\, dA(z)$.

So $\int_{\mathbb{D}} \|\Phi(z)\|_{\mathcal{F}} \, |h_1(z)| \, \|h_2'(z)\|_{\mathcal{F}} \ln\frac{1}{|z|}\, dA(z) \le \left(\int_{\mathbb{D}} \|\Phi(z)\|^2_{\mathcal{F}} |h_1(z)|^2 \ln\frac{1}{|z|}\, dA(z)\right)^{\frac{1}{2}}$

$\times \left(\int_{\mathbb{D}} \|h_2'(z)\|^2_{\mathcal{F}} \ln\frac{1}{|z|}\, dA(z)\right)^{\frac{1}{2}}$.

$\le (C C_1)^{\frac{1}{2}} \cdot C \|h_2\|_{H^2}$

(If $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$, then $\|f\|^2_{H^2_n} = \sum \|f_k\|^2_{H^2} = \sum |f_k(0)|^2 + \frac{2}{\pi} \int \underbrace{\sum |f_k'(z)|^2}_{\|f'(z)\|^2} \ln\frac{1}{|z|}\, dA$)

If $f \in \mathrm{Hol}(\Omega, \mathcal{H})$, $\partial\bar{\partial} \|f(z)\|^2_{\mathcal{H}} = \|f'(z)\|^2_{\mathcal{H}}$.

Forgotten (missed formula): If $f \in H^1$, then $\|f\|_{H^1} = |f(0)| + \frac{1}{2\pi} \int_{\mathbb{D}} \frac{|f'(z)|^2}{|f(z)|} \ln\frac{1}{|z|}\, dA(z)$.

If $f \in H^1_n$, then $\|f(0)\|_{\mathbb{C}^n} + \frac{1}{2\pi} \int_{\mathbb{D}} \frac{\|f'(z)\|^2_{\mathbb{C}^n}}{\|f(z)\|_{\mathbb{C}^n}} \ln\frac{1}{|z|}\, dA(z)$ is an equivalent norm uniformly in n.

(Corollary of Green's formula: $\frac{|f'(z)|^2}{|f(z)|} = \Delta |f(z)|$).

So $\int |\Phi| \cdot |h'| \ln\frac{1}{|z|}\, dA \le \underbrace{\left(\int |\Phi|^2 |h| \ln\frac{1}{|z|}\right)^{\frac{1}{2}}}_{\substack{\le C\|h\|^{\frac{1}{2}}_{H^1} \\ (\text{Carleson})}} \underbrace{\left(\int \frac{|h'|^2}{|h|} \ln\frac{1}{|z|}\, dA\right)^{\frac{1}{2}}}_{\substack{\le C\|h\|^{1/2}_{H^1} \\ (\text{forgotten formula})}}$