Estimates in Corona Theorem

If $\delta^2 \le \sum |f_n(z)|^2 \le 1$ (may have forgotten this upper bound the last time

then $\exists\ g_1 \cdots g_n \in H^\infty$ s.t. $\sum g_K f_K \equiv 1$ and $\left(\sum |g_K(z)|^2\right)^{1/2} \le C(\delta)$, where $C(\delta) \le \frac{C}{\delta^2} \ln \frac{e}{\delta}$. Nonlinearity is essential here.

We will give an example to show that $C(\delta)$ cannot be better than C/δ^2.

$n=2$, $f_1(z) = \delta z$, $f_2(z) = \dfrac{z^N - \delta}{1-\delta z^N}$, where N is some large number to be determined later. (example due to V. Tolokonnikov)

1. $|f_1(z)| + |f_2(z)| \geq \frac{\delta}{2}$

How do we prove this? Consider functions $\dfrac{\omega-\delta}{1-\delta\omega}$.

$\exists\ \varepsilon > 0$ such that $|\omega| < \varepsilon \Rightarrow \left|\dfrac{\omega-\delta}{1-\delta\omega}\right| > \dfrac{\delta}{2}$.

Pick N s.t. $\varepsilon^{1/N} \ge \frac{1}{2}$ $\left(\Leftrightarrow \quad \varepsilon \geq \left(\frac{1}{2}\right)^N\right)$.

If $|z| \ge \frac{1}{2}$, then $\Rightarrow |f_1(z)| \ge \frac{\delta}{2}$.

If $|z| < \frac{1}{2}$, then $\Rightarrow |z^N| < \left(\frac{1}{2}\right)^N \le \varepsilon \Rightarrow \left|\dfrac{z^N-\delta}{1-\delta z^N}\right| > \dfrac{\delta}{2}$ $\quad (z^N = \omega)$

Lemma 1 $f_1 g_1 + f_2 g_2 \equiv 1$, $g_1, g_2 \in H^\infty$. Then $\|g_1\|_{H^\infty} \geq \frac{C}{\delta^2}$.

$$\|g_2\|_\infty \ge \|g_2(0)\| = \frac{1}{|f_2(0)|} \quad \text{because } f_1(0) = 0$$
$$= \frac{1}{\delta}$$

Now $g_1 = \dfrac{1-f_2 g_2}{f_1} = \dfrac{1-f_2 g_2}{\delta z}$ because $f_1 = \delta z$

$$\text{so } \|g_1\|_\infty \ge \frac{1}{\delta}\left\| \frac{1-f_2 g_2}{z}\right\|_{L^\infty(\mathbb{T})} \ge \frac{1}{\delta}\left(\left\| \frac{f_2 g_2}{z}\right\|_{L^\infty(\mathbb{T})} - \underbrace{\left\|\frac{1}{z}\right\|_{L^\infty(\mathbb{T})}}_{=1}\right) \ge \frac{1}{\delta}\left(\frac{1}{\delta} - 1\right) \ge \frac{1/2}{\delta^2}$$

f_2 and z are unimodular (absolute value 1) on $\mathbb{T}$ — for suff. small δ

So this simple example shows that the estimate is nonlinear.

Problem of Ideals

$1 \le C\left(\sum |f_k(z)|^2\right)^{1/2} \Rightarrow 1 \in I_{f_1,\dots,f_n}$ (ideal generated by functions $f_1,\dots,f_n$).

Is it true that $|\tau(z)| \le \underset{\uparrow}{C}\left(\sum |f_k(z)|^p\right)^{1/p} \ \forall z \overset{?}{\Rightarrow} \tau \in I_{f_1 \cdots f_n}$ No

constant not necessary because you can put it in τ instead

where $I_{f_1 \cdots f_n} = \{h = \sum g_k f_k,\ g_k \in H^\infty\}$

Counterexample by Rao says <u>no</u>.

Is it true that $|\tau(z)| \le \left(\sum |f_k(z)|^p\right)^{1/p} \ \forall z \overset{?}{\Rightarrow} \tau^p \in I_{f_1 \cdots f_n}$ Yes for $p > 2$ No for $p \le 2$

Is it true that $|\tau(z)| \le \left[\left(\sum |f_k(z)|^2\right)^{1/2}\right]^{p} \overset{?}{\Rightarrow} \tau \in I_{f_1 \cdots f_n}$ No for $p \le 2$ Yes for $p > 2$

To get this conclusion, had to show that $C(\delta) \le \frac{c}{\delta^2} \ln \ln \frac{1}{\delta}$

so this destroyed $p = 2$.

Angles between S^* invariant spaces

$\cos \alpha = \sup\{|(x,y)| : x \in X, y \in Y, \|x\| = \|y\| = 1\}$.

$X \wedge Y = 0 \Leftrightarrow \underbrace{X + Y}_{\text{direct sum}}$ is closed $\Leftrightarrow \underbrace{\mathcal{P}_{X\|Y}}_{\text{skew projection of } X \text{ parallel to } Y} : X + Y \to X$ is well-defined and bounded and $\|\mathcal{P}_{X\|Y}\| = \frac{1}{\sin \alpha}$ $\alpha = X \wedge Y$

$\Updownarrow$

$\{x + y : x \in X, y \in Y\}$

$\mathcal{P}_{X\|Y}(x+y) = x,\ x \in X, y \in Y$. (definition of projection) note: not orthogonal projection

<u>Example</u> θ_1, θ_2 are two inner functions. When is $\underbrace{K_{\theta_1} \wedge K_{\theta_2}}_{\text{angle between}} > 0$?

$K_\theta = H^2 \underset{\substack{\uparrow \\ \text{minus}}}{\ominus} \underset{\substack{\uparrow \\ \text{theta}}}{\theta} H^2$

<u>Prop</u> $K_{\theta_1} \cap K_{\theta_2} \ne \{0\} \Leftrightarrow \exists$ inner $\tilde{\theta}$ (nontrivial) ($\tilde{\theta} \ne$ constant) s.t. $\theta_1/\tilde{\theta},\ \theta_2/\tilde{\theta} \in H^\infty$.

If $K = K_{\theta_1} \cap K_{\theta_2}$, $S^* K \subset K \Rightarrow K = K_{\tilde{\theta}}$.

<u>Thm</u> θ_1, θ_2 are inner. Then $K_{\theta_1} \wedge K_{\theta_2} > 0 \Leftrightarrow |\theta_1(z)| + |\theta_2(z)| \ge \delta > 0 \quad \forall z \in \mathbb{D}$

i.e. it happens iff they satisfy the Corona condition.