Thm $K_{\theta_1} \wedge K_{\theta_2} > 0$ iff θ_1, θ_2 satisfy Carleson-Corona condition, i.e.

$$\forall z \in \mathbb{D}, \quad |\theta_1(z)| + |\theta_2(z)| \geq \delta > 0.$$

Pf $K = \mathrm{Cl}(K_{\theta_1} + K_{\theta_2})$

We know $S^* K \subset K$ (invariant subspace, being the sum of two such)

Also $K \neq H^2$ $(\theta_1 \theta_2 H^2 \perp K)$ since it is orth to each term

$\Rightarrow K = K_\theta$ $\left(\theta = \theta_1 \theta_2 / \mathrm{GCID} \right.$ ~ greatest common inner divisor $)$

Let $\mathcal{P} := \mathcal{P}_{K_{\theta_1} \| K_{\theta_2}}$, $T = S^*|_{K_\theta}$

$\Rightarrow TP = PT$

since $P(f_1 + f_2) = f_1$ $\quad TP(f_1+f_2) = S^* f_1$

$$S^*(f_1 + f_2) = S^* f_1 + S^* f_2$$

$$P(S^* f_1 + S^* f_2) = S^* f_1$$

$\exists \varphi \in H^\infty$ $\|\varphi\|_\infty = \|P\|$

s.t. $Pf = P_+(\bar{\varphi} f)$ $\forall f \in K_\theta$ by CLT.

If $f \in K_{\theta_1}$ $P_+(\bar{\varphi} f) = f$. Since 1 is symbol of identity operator,

then $\varphi = 1 + \theta_1 h$ $h \in H^\infty$.

If $f \in K_{\theta_2}$ $P_+(\bar{\varphi} f) = 0$. $\Rightarrow \varphi = 0 + \theta_2 h_2$.

$\Rightarrow \exists h_1, h_2 \in H^\infty$ s.t. $\theta_1 h_1 + \theta_2 h_2 = 1$, which implies the CC condition.

$\underline{\text{Rk}}$ $$\|\mathcal{P}_{K_{\Theta_1}\|K_{\Theta_2}}\| = \inf\{\|h_2\|_\infty : \quad h_1\Theta_1 + h_2\Theta_2 \equiv 1\}$$ (Bezout equation)

On the other hand,

$$\|\mathcal{P}_{K_{\Theta_2}\|K_{\Theta_1}}\| = \inf\{\|h_1\|_\infty : \quad " \quad " \quad \}$$

We always have $\|\mathcal{P}_{X\|Y}\| = \|\mathcal{P}_{Y\|X}\|$. So if Θ_1, Θ_2 inner, the two infima above are equal.

Invertibility of model operator

Let $T = S^*|_{K_\Theta}$, $A : K_\Theta \to K_\Theta$, $AT = TA$.

By CLT, $\exists \varphi \in H^\infty$ s.t $Af = P_+(\bar{\varphi} f)$ $\forall f \in K_\Theta$

A invertible $\iff A^{-1}$ exists and is bounded $(A^{-1}T = TA^{-1})$

$\iff \exists \psi \in H^\infty$ s.t. $A^{-1}f = P_+(\bar{\psi} f)$ $\forall f \in K_\Theta$.

If $f \in K_\Theta$, then $f = AA^{-1}f = P_+(\bar{\varphi} P_+(\bar{\psi} f))$

$$= P_+(\bar{\varphi}(P_+(\bar{\psi} f) + \underbrace{P_-(\bar{\psi} f)}_{\text{orthogonal to } H^2})) \qquad = P_+(\bar{\varphi}\bar{\psi} f)$$

$\Rightarrow$ $\varphi\psi = 1 + \Theta h$ i.e., $\varphi\psi$ is symbol of identity operator.

So A is invertible $\iff \exists \psi, h \in H^\infty$ s.t. $\varphi\psi - \Theta h \equiv 1$

$\iff |\varphi(z)| + |\Theta(z)| \geq \delta > 0$ $\forall z \in \mathbb{D}$ (CC condition).

Applying this to the special case $A = T = S^* | K_\theta$,

we have $\varphi(z) = z$, and so

$$\|T^{-1}\| = \inf\{\|h_1\|_\infty : h_1 z + h_2\theta \equiv 1,\ h_1, h_2 \in H^\infty\}$$

$$= \inf\{\|h_2\|_\infty : \quad " \quad " \quad " \quad \} = \frac{1}{|\theta(0)|},$$

(Exercise)

since $\|h_2\|_\infty \geq |h_2(0)| = \dfrac{1}{|\theta_2(0)|}$

and taking $h_2 = \dfrac{1}{\theta(0)}$, $h_1 = \dfrac{1 - \theta(z)/\theta(0)}{z} \in H^\infty$,

we have $zh_1 + \theta h_2 \equiv 1$.

Spectrum and Condition Number

If $A: \mathbb{C}^n \to \mathbb{C}^n$, then A is invertible if it has no zero eigenvalues,

Condition number measures "how invertible" A is.

$$\underbrace{\|A\|\,\|A^{-1}\|}_{\text{Condition number}} \sim \operatorname{dist}(0, \underbrace{\sigma(A)}_{\text{spectrum of } A})$$

If $\|A\| = 1$ then $\|A^{-1}\| \leq (\operatorname{dist}(0, \sigma(A)))^{-n}$, since

$$\|A^{-1}\| = s_n^{-1} \quad (s_n = \text{smallest singular value})$$

$$\leq (s_1 s_2 \cdots s_n)^{-1} = \det(A^*A)^{-1/2} = |\det A|^{-1}$$

$$= |\lambda_1 \cdots \lambda_n|^{-1} \leq |\lambda_n|^{-n} \quad \text{where } |\lambda_1| \geq \cdots \geq |\lambda_n|.$$

Rk: This seems like a rough estimate, but in fact it is sharp, i.e. if $|\lambda_n| = \delta$, $\|A^{-1}\| \leq \delta^{-1}$.

Rk $\|A\| = 1$ is just a scaling convention.

If $T = S^*|_{K_\theta}$, $\theta = \dfrac{z^n - \delta^n}{1 - \delta^n z}$ then $\|T^{-1}\| = |\theta(0)|^{-1} = \dfrac{1}{\delta^n}$