4/15/2009 COMPLEX ANALYSIS II

$T = S^*|_{K_\theta}$

If $\dim K_\theta > 1$, then $\|T\| = 1$.

We know $\|S^*\| = 1 \Rightarrow \|T\| \le 1$.

If $\dim K_\theta > 1$, the $\exists f \in K_\theta$ s.t. $\hat{f}(0) = 0$. So $Tf = S^* f = \frac{f}{z} \Rightarrow \|Tf\| = \|f\|$.

This is because $\{f \mid \hat{f}(0) = 0\}$ is codim 1.

Functions of operators (matrices)

For polynomials, this is clear. $p(z) = \sum_0^n a_k z^k$. We can define $p(A) = \sum_{k=0}^n c_k A^k$

Also exponentials. The sol'n to the linear ODE $\dot{x} = Ax$ is $x(t) = e^{tA} x_0$ where $e^{tA} = \sum_{n \ge 0} \frac{A^n}{n!}$

Similarly we can do that for rational function, analytic functions.

Riesz-Dunford Calculus

$A: X \to X$ operator $\sigma = \sigma(A)$ its spectrum.

Let $\Omega \underset{\text{open}}{\supset} \sigma$. Suppose $\varphi \in \mathrm{Hol}(\Omega)$

We want to define $\varphi(A)$.

Let γ be a contour in Ω around σ. Take bdd $\mathcal{U}$ open. $\sigma \subset \mathcal{U} \subset\subset \Omega$ compactly contained.

Let $\gamma_0 = \partial \mathcal{U}$. We can say γ is homotopically equivalent to γ_0 in $\Omega - \sigma$.

By construction, the winding number at any point in the spectrum is 1.

Define: $\varphi(A) = \int_\gamma \varphi(z)(z-A)^{-1} \frac{dz}{2\pi i}$ This not depend on choice of γ.

If A is a complex number, this is just Cauchy formula.

We want to show this def. makes sense.

1. If $\varphi \equiv 1$, $\varphi(A) = 1$
2. If $\varphi(z) = z$, $\varphi(A) = A$
3. If $\psi = \varphi_1 \varphi_2$, $\psi(A) = \varphi_1(A)\varphi_2(A)$

Linearity is clear from the definition. (for fixed A).

Pf: 1.) $\varphi(A) = \int_\gamma (z-A)^{-1} \frac{dz}{2\pi i}$. Let $\gamma =$ circle of rad r where $r > \|A\|$.

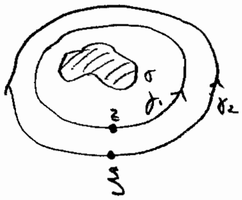
Then $(z-A)^{-1} = z^{-1}\left(1 - \frac{A}{z}\right)^{-1} = z^{-1} \sum_{n \ge 0} \left(\frac{A}{z}\right)^n$

So $\varphi(A) = \int_\gamma \sum A^n z^{-n-1} \frac{dz}{2\pi i} = \sum A^n \int_\gamma z^{-n-1} \frac{dz}{2\pi i}$ uniform convergence

$$= A^0 = 1.$$

2.) If $\varphi(z) = z$, the same calculation produces $\varphi(A) = A$.

3.)



$\sigma(A) \subset U_1 \subset\subset U_2 \subset\subset \Omega$

$\gamma_k = \partial U_k$

$$\varphi_1(A)\varphi_2(A) = \int_{\gamma_1} \varphi_1(z)(z-A)^{-1} \frac{dz}{2\pi i} \int_{\gamma_2} \varphi_2(\xi)(\xi - A)^{-1} \frac{d\xi}{2\pi i}$$

$$= \iint_{\gamma_1 \gamma_2} \varphi_1(z)\varphi_2(\xi) \underbrace{(z-A)^{-1}(\xi-A)^{-1}}_{= \frac{1}{\xi - z}\left[(z-A)^{-1} - (\xi-A)^{-1}\right]} \frac{d\xi}{2\pi i}\frac{dz}{2\pi i}$$

$$= \underbrace{\iint_{\gamma_1 \gamma_2} \varphi_1(z)\varphi_2(\xi) \frac{1}{\xi - z}(z-A)^{-1} \frac{dz}{2\pi i}\frac{d\xi}{2\pi i}}_{= \int_{\gamma_1} \varphi_1(z)\varphi_2(z)(z-A)^{-1}\frac{dz}{2\pi i}} - \underbrace{\iint_{\gamma_1 \gamma_2} \varphi_1(z)\varphi_2(\xi)\frac{1}{\xi - z}(\xi - A)^{-1}\frac{dz}{2\pi i}\frac{d\xi}{2\pi i}}_{= 0}$$

0 since $\frac{1}{\xi - z}$ is analytic function inside γ_1 as a function of z

$$= \int_{\gamma_1} \psi(z)(z-A)^{-1}\frac{dz}{2\pi i} = \psi(A). \square$$

<u>Thm (Spectral Mapping Theorem)</u> $\sigma(\varphi(A)) = \varphi(\sigma(A))$

<u>Thm:</u> If $\psi = \varphi_1 \circ \varphi_2$, then $\psi(A) = \varphi_1(\varphi_2(A))$

<u>Example:</u> Let A be an $N \times N$ matrix. Let $\lambda_1, \ldots \lambda_n$ $(n \le N)$ be the eigenvalues.

Let $\varphi_k(z) = \begin{cases} 1 & \text{if in a neighborhood of } \lambda_k \\ 0 & \text{otherwise} \end{cases}$

$\varphi_k(A) := \frac{1}{2\pi i}\int_{\gamma_k} (z-A)^{-1} dz$. Note $\varphi_k(A)^2 = \varphi_k(A)$. So $\varphi_k(A)$ is a projection.

And of course, $\sum \varphi_k(A) = 1$. In fact, $\varphi_k(A) =$ projection onto λ_k-generalized eigenspaces.

From this, you can get the Jordan decomposition very easily.