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1. "Forgotten formula" $\|f\|_{H^1} \asymp |f(0)| + \iint_{\mathbb{D}} \frac{|f'(z)|^2}{|f(z)|} \ln \frac{1}{|z|}\, dA$

for vector-valued functions

write down all the details

2. p. 305 #3 interpolation with derivatives, not very hard

3. p. 308 # 15

4. Carleson's proof of Corona Theorem. Two parts:
   1. Reduction to the case of two Blaschke products
   2. Construction of "Carleson contour."

      Best construction of the contour is done by Burgain, "on finitely generated ideals of H^∞"

      Very cool construction, but there is a mistake; problem is to find the mistake.

5. p. 270 # 16 Characterization of BMO in terms of commutators

6. p. 264 # 9 : It is not true that $f \in H^\infty \Rightarrow |f'|\, dx\, dy$ is Carleson. (construct counterexample)

   If it were true, it would simplify the Corona theorem and many things.

7. p. 170 #4 : badly-approximable functions (by H^∞)

   distance to H^∞ from a bdd fcn is given by Hankel operator

   if it is compact, the distance is always attained.

   badly-approximable functions are those where the distance to H^∞ is the norm of the function, so the best approximation is 0.

8. p. 173 # 19 Classical, about extreme points in H^∞ in the unit ball

9. p. 89 # 9

10. p. 44 #8.

11. Reproducing Kernel thesis for Hankel operators:

    If $k_\lambda = \dfrac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda} z}$, $\|\Gamma k_\lambda\| \le A \Rightarrow \|\Gamma\| \le CA$.

    interesting to see what best constant C you get; we can get $2\sqrt{e}$.

    have to check it on the unit ball in infinite-dimensional space.

    related to BMO and to Garsia's norm.

12. See next page →

12\.

Estimates in Corona theorem

for disk algebra $A(\mathbb{D}) = H^\infty \cap C(\mathbb{T})$

If $1 \le \sum |f_k(z)|^2 \ge \delta^2 > 0 \Rightarrow \exists\, g_k$ with $\sum g_k f_k \equiv 1$ and $\sum |g_k(z)|^2 \le C(\delta,$

Show that if we have estimate $C_{H^\infty}(\delta)$ in the H^∞ problem, then

for any $\varepsilon > 0$, $C_{H^\infty}(\delta) + \varepsilon$ will be an estimate in the disk algebra.

So, the estimate in the disk algebra is the same as the estimate in the Corona theorem.

$$C_{\mathcal{A}}(\delta) = \sup_{\substack{f_k \in \mathcal{A} \\ 1 \ge \sum |f_k(z)|^2 \ge \delta > 0}} \ \inf_{\substack{g_k \in \mathcal{A} \\ \sum g_k f_k \equiv 1}} \ \sup_{z \in \mathbb{D}} \left(\sum |g_k(z)|^2 \right)^{1/2}$$

Show that $C_{A(\mathbb{D})}(\delta) = C_{H^\infty}(\delta)$.

13\. Is $C_{H^\infty}(\delta)$ a continuous function of δ?

It is true, but it does not follow from general abstract reasoning.

14\. Reproducing Kernels and Calabi Rigidity

Analyticity: If you have two reproducing kernels $K_1(z,w)$ and $K_2(z,w)$ and you know

$K_1(z,z) = K_2(z,z)$ for all z (kernels coincide on the diagonal) then $\Rightarrow K_1 = K_2$.

has to be an open manifold, can't be compact

this is a local statement

Stationary processes:

Toeplitz determinants

Bellman function and Hunt-Muckenhoupt ← (if you like playing with matrices)

Wheeden theorem

no specifics on these because we will discuss them in the next couple of days

Applications to perturbation theory, Kato-Rosenblum theorem

In the next week or so, we will do a topic related to probability

Stationary (Gaussian) processes

what is stationary process in discrete time?

sequence $\{\xi_n\}_{n=-\infty}^{\infty}$ $n =$ time, ξ random variable (not independent, but identically distributed)

Stationary means that if we take any finitely many $(\xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_n})$

joint distribution functions on these two coincide $(\xi_{k_1+N}, \xi_{k_2+N}, \dots, \xi_{k_n+N})$

expectation

$\mathbb{E}\, \xi_k \overline{\xi_n} = d(n-k)$ depends only on the difference

assume $\mathbb{E}(\xi_n) = 0$.

General model:

$L^2(\Omega, dP) \ni \xi_n$ with inner product $(\xi, \eta) = \mathbb{E}(\xi \bar{\eta})$

space; measure is your probability

$\mathcal{H}$ $\{\xi_n\}_{n=-\infty}^{\infty}$ stationary means that $(\xi_k, \xi_n) = d(n-k)$ depends only on difference

Example of stationary process:

- nonnegative measure μ on $\mathbb{T}$, consider $L^2(\mu)$ (space of square-integrable functions on circle)

$$\xi_n = z^n, \quad (\xi_n, \xi_k) = \int_{\mathbb{T}} z^n z^{-k}\, d\mu(z) = \int_{\mathbb{T}} z^{n-k}\, d\mu(z) = \hat{\mu}(k-n)$$

Fourier coefficient of measure μ with numbers $k-n$.

$\hookrightarrow$ In fact, those are the only example of a stationary sequence.

μ is called a spectral measure process

d is a complex-valued function depending on a real argument

If we consider a map $L^2(\mu) \longrightarrow L^2(\Omega, dP)$

$z^n \longmapsto \xi_n$ then it is an isometry.

So instead of working on a probability space, work with a system of exponentials in L^2 (something we know, because probability space is often incomprehensible).