Prediction & Szegö theorem.

$\{\xi_n\}$, $P = \mathcal{L}(\xi_n, n<0)$.

$\mathrm{dist}(\xi_0, P)^2 = \inf \mathbb{E}|\xi_0 - \sum_{k<0} c_k \xi_k|^2$.

$\mathrm{dist}_{L^2(\mu)}(1, \mathcal{L}(z^n, n<0)) = \mathrm{dist}_{L^2(\tilde{\mu})}(1, \mathcal{L}(z^n, n>0))$ where $\tilde{\mu}(E) = \mu(\bar{E})$.

Thm. (Szegö)

Let $d\mu = w \frac{|dz|}{2\pi} + \underset{\text{some singular part.}}{\underline{\underline{\mu_s}}}$

Then, $\mathrm{dist}_{L^2(\mu)}(1, \mathcal{L}(z^n, n>0)) = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \ln w \frac{|dz|}{2\pi}\right) = \Delta_w = \Delta_\mu$.

pf. ① Let $d\mu = w\frac{|dz|}{2\pi}$; no singular part.

$\int \ln w > -\infty$.

Then, $\exists\, h \in H^2$, $|h|^2 = w$ a.e. on $\mathbb{T}$.

$h(z) = \exp\left(\frac{1}{2}\int \frac{\zeta+z}{\zeta-z} \ln w(\zeta) \frac{|d\zeta|}{d\pi}\right)$.

- $\mathrm{dist}_{L^2(\mu)}(1, \mathcal{L}(z^n, n>0))$

$= \inf\{\int|1-zp|^2 w \frac{|dz|}{2\pi} ;\ \underset{\text{polynomial.}}{p} \in \mathcal{L}(z^n, n\geq 0)\}$

$\Delta\tilde{w} = \Delta w$, $\tilde{w}(z) = w(\bar{z})$.

$\int_{\mathbb{T}} |1-zp|^2 w(z) \frac{|dz|}{2\pi} = \int_{\mathbb{T}} |\underbrace{h-zph}_{\hookrightarrow F}|^2 \frac{|dz|}{2\pi}$ $(\leftarrow w = |h|^2)$

$= \sum_{k=0}^{\infty} |\hat{F}(k)|^2 = |h(0)|^2 + \sum_{k=1}^{\infty}|\hat{F}(k)|^2$

$= |h(0)|^2 + \underbrace{\int_{\mathbb{T}} |h-h(0)-zph|^2 \frac{|dz|}{2\pi}}_{\geq 0}$.

$|h(0)| = \Delta_w$ (by definition).

$\Rightarrow \boxed{\mathrm{dist}(\cdots) \geq \Delta_w}$

- h is outer $\Rightarrow \{hp,\ p \in \mathcal{L}(z^n, n\geq 0)\}$ dense in H^2.

$p_k \in \mathcal{L}(z^n, z\geq 0)$. $p_k h \to \frac{h-h(0)}{z}$ in L^2

$zp_k h \to h - h(0)$ in L^2

$$\int_{\mathbb{T}} |h - h(0) - z p_k h|^2 \frac{|dz|}{2\pi} \to 0.$$

$$\Rightarrow \boxed{\mathrm{dist}(\cdots) \le \Delta_w}$$

<u>Lemma.</u> If $d\mu = w \frac{|dz|}{2\pi} + \mu_s$, $w \ge \delta > 0$,

then $\mathrm{dist}_{L^2(\mu)}(1, \mathcal{L}(z^n, n \ge 1)) = \Delta_w$.

<u>proof.</u> $d\mu = w \frac{|dz|}{2\pi} + \mu_s$, $\quad \mu_\delta = (w+\delta)\frac{|dz|}{2\pi} + \mu_s$.

$\forall f \in \mathcal{L}(z^n ; n \ge 0)$, $\|f\|^2_{L^2(\mu)} \le \|f\|^2_{L^2(\mu_\delta)}$.

$$\Rightarrow \underbrace{\mathrm{dist}_{L^2(\mu)}(1, \mathcal{L}(z^n ; n \ge 1))}_{=: d(\mu)} \le \mathrm{dist}_{L^2(\mu_\delta)}(1, \mathcal{L}(z^n ; n \ge 1))$$

Take $\varepsilon > 0$. $p \in \mathcal{L}(z^n ; n \ge 1)$ s.t. $\|1 - p\|_{L^2(\mu)} \le d(\mu) + \frac{\varepsilon}{2}$.

$$\because \lim_{\delta \to 0} \|1 - p\|_{L^2(\mu_\delta)} = \|1 - p\|_{L^2(\mu)}$$

$\Rightarrow \exists \delta_0$ s.t. $\forall \delta < \delta_0$, $0 \le \|1 - p\|_{L^2(\mu_\delta)} - \|1 - p\|_{L^2(\mu)} < \frac{\varepsilon}{2}$.

So, $\|1 - p\|_{L^2(\mu_\delta)} \le d(\mu) + \varepsilon$.

$\Rightarrow d(\mu) \le d(\mu_\delta) \le d(\mu) + \varepsilon$.

$$\text{So, } d(\mu) = \lim_{\delta \to 0+} d(\mu_\delta) = \lim_{\delta \to 0+} \exp\left[\frac{1}{2}\int \ln(w+\delta) \frac{|dz|}{2\pi}\right]$$

$$= \exp\left[\frac{1}{2}\lim_{\delta \to 0+}\int \ln(w+\delta) \frac{|dz|}{2\pi}\right] = \exp\left(\frac{1}{2}\int \ln w \frac{|dz|}{2\pi}\right). \quad ⊞$$

1. How to find a minimizer.

1. $d\mu = w \frac{|dz|}{2\pi} + \mu_s$, $\Delta_w > 0$. $\quad w = |h|^2$ a.e. on $\mathbb{T}$. h : outer $\in H^2$.

$$\int |h - zph|^2 \frac{|dz|}{2\pi} \to \min.?$$

$$zp = \frac{h - h(0)}{h} = \sum c_k z^k$$