Finishing the proof of Szegö Thm:

Why doesn't μ_s part matter?

$d\mu = w\frac{|dz|}{2\pi} + \mu_s.$

$$\|f\|_{L^2(w)} \le \|f\|_{L^2(\mu)} \implies \mathrm{dist}_{L^2(w)}(1, \mathcal{L}(z^n, n\geq 1)) \le \mathrm{dist}_{L^2(\mu)}(1, \mathcal{L}(z^n, n\geq 1)).$$

Want to prove: If $A \subset \mathbb{T}$, $|A|=0$, $\mu_s(A^c)=0$ ($|A|$ = Lebesgue measure of A).

then $f\mathbb{1}_A$ is approximated by polynomials $P_k \in \mathcal{L}(z^n, n\geq 1)$

$$\|f\mathbb{1}_A - P_k\|_{L^2(\mu)} \longrightarrow 0. \quad \forall f \in L^2(\mu).$$

$p \in \mathcal{L}(z^n, n\geq 1)$ s.t. $\|1-p\|_{L^2(w)} < \frac{\varepsilon}{2} + \Delta_w$

Let $f = (1-p)\mathbb{1}_A$

$\exists\, q \in \mathcal{L}(z^n, n\geq 1)$ s.t. $\|f - q\|_{L^2(\mu)} < \frac{\varepsilon}{2}$

$$\|1-p-q\|_{L^2(\mu)} \le \underbrace{\|(1-p)\mathbb{1}_A - q\|_{L^2(\mu)}}_{\le \frac{\varepsilon}{2} + \Delta_w} + \underbrace{\|(1-p)\mathbb{1}_{A^c}\|_{L^2(\mu)}}_{= \|(1-p)\mathbb{1}_{A^c}\|_{L^2(w)}}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \Delta_w = \varepsilon + \Delta_w.$$

<u>Lemma</u> (Kolmogorov-Wold decomposition).

U - isometry ($U^*U = I$) in some Hilbert space $\mathcal{H}$.

Then $\mathcal{H} = \mathcal{E}_1 \oplus \mathcal{E}_2$, where $\mathcal{E}_1 = \mathrm{span}(U^n E, n\geq 0) = \bigoplus_{n=0}^{\infty} U^n E$, $E = \mathcal{H} \ominus U\mathcal{H}$.

$$\mathcal{E}_2 = \bigcap_{n\geq 0} U^n \mathcal{H}$$

(closed linear span).

Moreover, $U\mathcal{E}_1 \subset \mathcal{E}_1$ and $U\mathcal{E}_2 \subset \mathcal{E}_2$, $U^*\mathcal{E}_2 \subset \mathcal{E}_2$.

Pf: $U\mathcal{E}_1 \subset \mathcal{E}_1$ clear

$U\mathcal{E}_2 \subset \mathcal{E}_2$ since $\mathcal{E}_2 = \bigcap_{n\geq k} U^n\mathcal{H} \implies U\mathcal{E}_2 \subset \mathcal{E}_2$.

Also, $U^* = U^{-1}$, so we see also that $U^*\mathcal{E}_2 \subset \mathcal{E}_2$.

$$\left(\begin{array}{l} U^* U^k \mathcal{H} = U^{k-1}\mathcal{H} \\ U^*\mathcal{E}_2 = U^* \bigcap_{n\geq k} U^n\mathcal{H} = \bigcap_{n\geq k} U^{n-1}\mathcal{H} = \mathcal{E}_2 \end{array} \right)$$

We know $E \perp U\mathcal{H}$ by definition.

On the other hand, $U^k E \subset U^k\mathcal{H} \subset U\mathcal{H}$ for $k\geq 1$.

So $U^k E \perp E$ for $k \geq 1$.

$\therefore$ $U^k E \perp U^{k+n} E$ for all $n\geq 1$, $\forall\, k\geq 0$.

Proof of lemma cont.

$$\left.\begin{array}{l} \text{If } x \perp \mathcal{E}_1, \text{ then } x \perp U^n E \ \forall n. \\ \mathcal{H} = E \oplus UE \oplus \ldots \oplus U^{n-1}E \oplus U^n \mathcal{H}. \end{array}\right\} \Rightarrow x \in \mathcal{E}_2 = \bigcap_{n \geq 0} U^n \mathcal{H}$$

Now set $\mathcal{H} = \text{span}_{L^2(\mu)}(z^n, n \geq 0)$, $Uf = zf$. (μ on $\mathbb{T} \Rightarrow U$ isometry $-$ $|z|=1$ on $\mathbb{T}$).

$E = \mathcal{H} \ominus U\mathcal{H}$

Let $f \in E$. Then $f \perp z^n f$ $\forall n \geq 1$.

$$\Rightarrow \int z^n f \bar{f} \, d\mu(z) = 0 \quad \forall n \geq 1.$$

$$\int \bar{z}^n |f|^2 \, d\mu(z) = 0 \quad \forall n \geq 1 \text{ also.}$$

So $|f|^2 d\mu = \frac{|dz|}{2\pi} \cdot \text{constant}$ (multiple of Lebesgue measure).

In particular, $f = 0$ μ_s almost everywhere. ($f=0$ with respect to μ_s a.e).

If $g \in E$, then $f \perp z^n g$, $z^n f \perp g$ $\forall n \geq 1$. ($E \perp U^n E$).

$$\left.\begin{array}{l} \Rightarrow \int f\bar{g}\,\bar{z}^n \, d\mu = 0 \\ \int f \bar{g} z^n \, d\mu = 0 \end{array} \quad \forall n \geq 1. \right\} \begin{array}{l} \Rightarrow f\bar{g}\, d\mu = \alpha \frac{|dz|}{2\pi}. \\ \Rightarrow g = \text{const.} f. \end{array}$$

So $\mathcal{E}_1 = \text{span}_{L^2(\mu)}(z^n f, n \geq 0)$.

$f = 0$ μ_s a.e, $f \neq 0$ a.e. (w.r.t Lebesgue measure)

$$\Rightarrow \left\{ f \in L^2(\mu) : \int |f|^2 w \frac{|dz|}{2\pi} = 0 \right\} \subset \mathcal{E}_2.$$

$$\text{span}_{L^2(\mu)} \bar{z}^n \mathcal{H} = \text{span}_{L^2(\mu)} \{ z^n, n \in \mathbb{Z} \} = L^2(\mu).$$

$$\text{span}_{L^2(\mu)} (\bar{z}^n \mathcal{E}_1, n \geq 0) = \text{span}_{L^2(\mu)} \{ z^n f : n \in \mathbb{Z} \}.$$

$$= \left\{ f \in L^2(\mu) : \int |f|^2 d\mu_s = 0 \right\}$$

$\mathcal{E}_2 \perp \mathcal{E}_1 \Rightarrow \mathcal{E}_2 \perp \bar{z}^n \mathcal{E}_1 \ \forall n$