4/24/2009 COMPLEX ANALYSIS II

Conclusion of Pf from last time

$\mathcal{H} = \mathcal{E}_1 \oplus \mathcal{E}_2$

$\mathcal{E}_2 = \bigcap_{n \geq 0} U^n \mathcal{H}$

$U \mathcal{E}_2 \subset \mathcal{E}_2$

$U^* \mathcal{E}_2 \subset \mathcal{E}_2$

$U \mathcal{E}_2 = \mathcal{E}_2 \quad \because U\mathcal{E}_2 = \bigcap_{n \geq 1} U^n \mathcal{H} = \mathcal{E}_2$

In the situation $Uf = zf$ in $L^2(\mu)$.

$\mathcal{E}_1 = \operatorname{span}_{L^2(\mu)} \{ z^n f \mid n \geq 0 \}$.

$|f|^2 d\mu = \text{const} \cdot \frac{|dz|}{2\pi}$

$f \neq 0 \ |dz|$ a.e.

$f = 0 \ d\mu_s$ a.e

We chose $A \subset \mathbb{T}$, $|A| = 0$, $\mu_s(A^c) = 0$.

$\mathcal{H} = \operatorname{span}_{L^2(\mu)} (z^n \mid n \geq 0)$

$\operatorname{span}_{L^2(\mu)} (z^{-n} \mathcal{H} \mid n \geq 0) = \operatorname{span}_{L^2(\mu)} (z^n \mid n \in \mathbb{Z}) = L^2(\mu)$ (Measure theoretic result: Continuous functions are dense in L^2 of a Borel measure).

$\operatorname{span}_{L^2(\mu)} \{ z^{-n} \mathcal{E}_1 \mid n \geq 0 \} = \operatorname{span}_{L^2(\mu)} \{ f z^n \mid n \in \mathbb{Z} \}$

$= \mathbb{1}_{A^c} L^2(\mu)$

We know $z^{-n} \mathcal{E}_2 = \mathcal{E}_2$ because $z \mathcal{E}_2 = \mathcal{E}_2$

So $z^{-n} \mathcal{H} = z^{-n} \mathcal{E}_1 \oplus z^{-n} \mathcal{E}_2 = z^{-n} \mathcal{E}_1 \oplus \mathcal{E}_2$ $\quad \oplus$ = orthogonal direct sum

Thus $\mathcal{E}_2 = (\mathbb{1}_{A^c} L^2(\mu))^{\perp} = \mathbb{1}_A L^2(\mu)$

$z^{-1} \mathbb{1}_A$ can be approximated by $\mathcal{L}(z^n \mid n \geq 0)$ since $z^{-1} \mathbb{1}_A \in \mathcal{E}_2 \subset \mathcal{H}$.

Thus $\mathbb{1}_A$ can be approximated by $\mathcal{L}(z^n \mid n \geq 1)$. $\square$

Applications of Szegö Theorem

1. Asymptotics of orthogonal polynomials.

μ a positive measure on $\mathbb{T}$. So we have $L^2(\mu)$.

Apply Gram-Schmidt to $1, z, z^2, \dots$

Let φ_n = orthogonal polynomial of degree n.

$$\int_{\mathbb{T}} \varphi_n(z)\, \bar{z}^k \, d\mu(z) = 0 \quad \forall k,\ 0 \le k < n.$$

Normalization: Assume $\varphi_n(z)$ is monic. So $\varphi_n(z) = z^n + \dots$

$\varphi_n(z)$ solves minimization problem

$$\min \left\{ \|p\|_{L^2(\mu)} \mid p \in \mathcal{L}(z^k, 0 \le k \le n) \mid p(z) = z^n + \dots \right\}.$$

$$\min \left\{ \|p\|_{L^2(\mu)} \mid p \in \mathcal{L}(z^k, 0 \le k \le n\}, p(0) = 1 \right\} \quad \text{is attained}$$

at $\varphi_n^{\#}(z) = z^n \overline{\varphi_n\left(\frac{1}{\bar z}\right)}$ If $\varphi_n = \sum_{k=0}^{n} a_k z^k$, $\varphi_n^{\#} = \sum_{k=0}^{n} \overline{a_{n-k}}\, z^k$

We have the following recurrence: $\varphi_{n+1} = z\varphi_n + \alpha_n \varphi_n^{\#}$

Asymptotics of φ_n $(\varphi_n^{\#})$

$$\|\varphi_n\|_{L^2(\mu)} = \|\varphi_n^{\#}\|_{L^2(\mu)} \longrightarrow \operatorname{dist}_{L^2(\mu)}\left(1, \mathcal{L}(z^n, k \ge 1)\right) = \Delta_w$$

$\|\varphi_n^{\#}\|_{L^2(\mu)} = \operatorname{dist}(1, \mathcal{L}(z^k, 1 \le k \le n))$

$\Delta_w = \exp\left(\frac{1}{2}\int_{\mathbb{T}} \ln w \frac{|dz|}{2\pi}\right)$

w = absolutely continuous part of μ.

If $w = |h|^2$ a.e. on $\mathbb{T}$ where h outer $h(0) > 0$. Then $\dfrac{\varphi_n^{\#}(z)}{\|\varphi_n^{\#}\|_{L^2(\mu)}} \to h^{-1}$ universally on compact subsets.

2. Asymptotics of Toeplitz determinants.

$d\mu = w \frac{|dz|}{2\pi} + d\mu_s$ Construct finite Toeplitz matrices $T_n(\mu) = \begin{pmatrix} \hat\mu(0) & \hat\mu(-1) & \cdots & \hat\mu(-n) \\ \hat\mu(1) & \hat\mu(0) & & \\ \vdots & & \ddots & \\ \hat\mu(n) & & & \hat\mu(0) \end{pmatrix}$ $(n+1)\times(n+1)$ matrix

Asymptotics of $\det T_n(\mu)$?

Thm: (weak Szegö asymptotics)

$$\left[\det T_n(\mu)\right]^{1/n+1} = \Delta_w^2 + o(1) \quad \text{as } n \to \infty.$$

Lemma: $\det T_n(\mu) / \det T_{n-1}(\mu) = \operatorname{dist}_{L^2(\mu)}\left(z^n, \mathcal{L}(z^k \mid 0 \le k < n)\right)^2 = \operatorname{dist}_{L^2(\mu)}\left(1, \mathcal{L}(z^k \mid 1 \le k \le n)\right)^2$

Lemma: $\vec{x}_0 \dots \vec{x}_n$ l.i. vectors in some Hilbert space

Let $G_n = \langle\langle \vec{x}_i, \vec{x}_j \rangle\rangle$ = matrix of inner products. Similarly G_{n-1}. Then $\det G_n / \det G_{n-1}$ $= \operatorname{dist}(\vec{x}_n \mid \mathcal{L}(\vec{x}_k \mid k < n))$

Hint: Use row reduction to compute the determinants.