Helson - Szegö Thm

We have $L^2(\mu)$ $\mu \geq 0$ on $\mathbb{T}$

$$\mathcal{P} = \operatorname{span}_{L^2(\mu)}(z^n : n < 0)$$

$$\mathcal{F} = \operatorname{span}_{L^2(\mu)}(z^n : n \geq 0)$$

We are interested in the case $\mathcal{P} \wedge \mathcal{F} > 0$.

This occurs iff P^+ is bounded on $L^2(\mu)$.

<u>Def</u> $\{x_n\}_1^\infty$ in X is a basis if $\forall x \in X$ $\exists! \{c_k\}$ s.t. $x = \sum_1^\infty c_k x_k$.

(where the infinite sum is defined as $\lim_{n\to\infty} \sum_1^n c_k x_k$)

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① $\{x_k\}_1^\infty$ is a basis $\Rightarrow$ $\{x_k\}$ is total (i.e. $\mathcal{L}(x_k : k \geq 1)$ dense in X).

② " " $\Rightarrow$ $\{x_k\}$ is linearly independent (LI)

Given $\{x_k\}$ LI define P_n on $\mathcal{L}(x_k : k \geq 1)$ by

$$P_n \sum_1^\infty c_k x_k = \sum_1^n c_k x_k.$$

Thm (Banach Basis Thm)

Suppose $\{x_n\}$ total and LI. Then $\{x_n\}$ is a basis iff $\|P_n\| \le C < \infty$.

Suppose P_n are bounded operators in X. Then ① ⇔ ② and ② ⇒ ③ where

① $\{x_n\}$ is a basis

② $P_n x \to x \quad \forall x$

③ $\|P_n\| \le C < \infty$ by uniform boundedness

If $\|P_n\| \le C < \infty$ and $P_n x \to x$ $\forall x \in \mathcal{L}(x_k : k \ge 1)$ then $P_n x \to x \quad \forall x$ by $\varepsilon/3$ thm.

Thm $L^2(\mu)$ TFAE:

① $\mathcal{P}^\wedge \mathcal{F} > 0$

② P_+ is bounded in $L^2(\mu)$
$$\left(P_+ f = \sum_{k \ge 0} \hat{f}(k) z^k \right)$$

③ $\|P_{k,n}\|_{L^2(\mu)} \le C < \infty$ where $P_{k,n} = \sum_{j=k}^{n} \hat{f}(j) z^j$ $\forall k, n \in \mathbb{Z}$.

④ Hilbert Transform T is bounded in $L^2(\mu)$.

⑤ $\{z^n\}_{-\infty}^{\infty}$ is a basis in $L^2(\mu)$.

Proof

① $\Rightarrow$ ② Trivial

② $\Rightarrow$ ③ $P_{k,n} f = z^k P_+(\bar{z}^k f) - z^{n+1} P_+(z^{-n-1} f)$

③ $\Rightarrow$ ② $P_+ f = \lim_{n\to\infty} P_{0,n} f \quad \forall f \in \mathcal{L}(z^n : n \in \mathbb{Z})$.

If $\|P_{0,n}\| \le C \Rightarrow P_{0,n} f \to P_+ f$

$\forall f \in L^2(\mu)$ by "$\varepsilon/3$ - theorem"

(② & ③) $\Rightarrow$ ④ $T = i(I - P_+) - i(P_+ - P_{0,0})$

⑤ $\Leftrightarrow$ ③ (Banach Basis Thm)

④ $\Rightarrow$ ② (exercise)

Thm (Helson - Szegö Thm)

P_+ is bounded in $L^2(\mu)$ iff μ is absolutely continuous $\left(d\mu = w \frac{|dz|}{2\pi}\right)$ where $w = \exp(u + \tilde{v})$ for some $u, v \in L^\infty$, $\|v\|_\infty < \frac{\pi}{2}$, and $\tilde{v}$ is the Hilbert transform of v.

(Recall the Hilbert transform of f is

$$Tf = i \sum_{k<0} \hat{f}(k) z^k - i \sum_{k>0} \hat{f}(k) z^k.)$$

Proof

① P_+ bounded $\Rightarrow$ $d\mu = w \frac{|dz|}{2\pi}$ (μ absolutely continuous)

since if $\mu_s \neq 0$, $A \subset \mathbb{T}$, $|A| = 0$, $\mu_s(A^c) = 0$

then $1_A L^2(\mu) = \mathcal{E}_2 \subset \text{span}_{L^2(\mu)}(z^n : n \geq 0)$

$\subset \text{span}_{L^2(\mu)}(z^n : n \geq 1)$.

and $1_A L^2(\mu) \subset \text{span}_{L^2(\mu)}(\bar{z}^n : n \geq 1)$.

so if $\mu_s \neq 0$ then $\underset{\text{"past"}}{\mathcal{P}} \cap \underset{\text{"future"}}{\mathcal{F}} \neq \{0\}$

② If $\{z^n\}_{-\infty}^{\infty}$ is a basis in $L^2(\mu)$ then

$\text{dist}_{L^2(\mu)}(1, \mathcal{L}(z^n : n \geq 1)) > 0 \Rightarrow \int_{\mathbb{T}} \ln w > -\infty$.

③ $\exists\, G$ outer s.t. $|G|^2 = w$ a.e..

$$\sup_{\substack{f \in \mathcal{F} \\ g \in \mathcal{P} \\ \|f\|, \|g\| \leq 1}} \int_{\mathbb{T}} f\bar{g}\, \underset{|G|^2}{w} \frac{|dz|}{2\pi} = \sup \int \underbrace{Gf}_{f_1}\, \underbrace{\overline{\bar{G}g}}_{g_1} \frac{|G|^2}{G^2} \frac{|dz|}{2\pi},$$

G is outer, so $\{Gf : f \in \mathcal{L}(z^n : n \geq 0)\}$ is dense in H^2, and $\{\bar{G}g : g \in \mathcal{L}(z^n \cdot n < 0)\}$ is dense in H^2_-.

$$\|f_1\|^2_{H^2} = \|Gf\|^2_2 = \|f\|_{L^2(\omega)} = \int_{\mathbb{T}} |f|^2 |G|^2 \frac{|dz|}{2\pi}$$

$$\|g_1\|^2_{H^2} = \|g\|_{L^2(\omega)} = \sup_{\substack{f_1 \in H^2 \\ g_1 \in H^2_- \\ \|g_1\|_2,\ \|f_1\|_2 \le 1}} \left(\frac{|G|^2}{G^2} f, g \right)_{L^2} = \underbrace{\left\| H_{\frac{|G|^2}{G^2}} \right\|}_{\text{Hankel operator}}$$

So $P^\wedge \mathcal{F} > 0$ iff $\left\| H_{\frac{|G|^2}{G^2}} \right\| < 1$

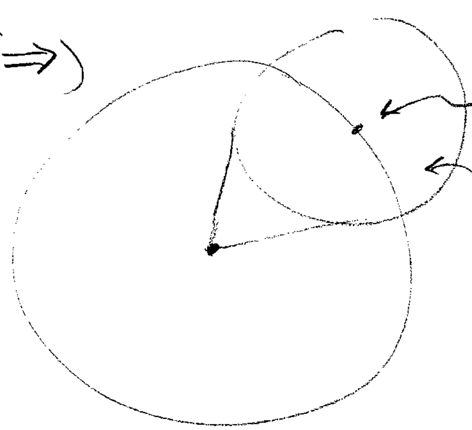
iff $\exists h \in H^\infty$ s.t. $\left\| \frac{|G|^2}{G^2} - h \right\|_\infty < 1$

(by Nehari)

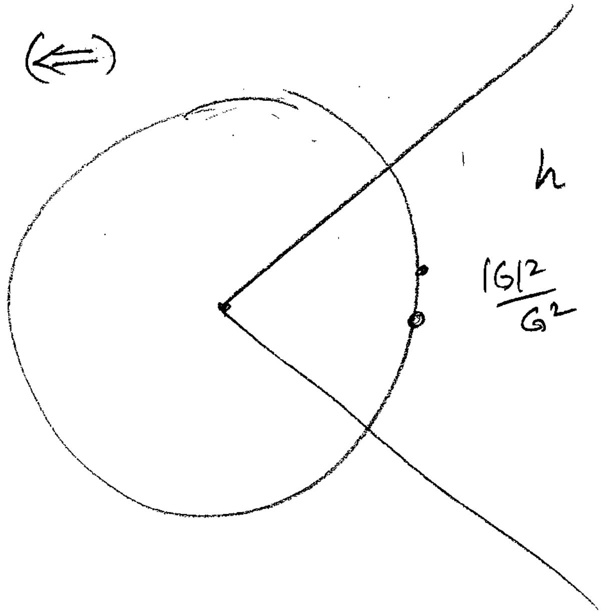
iff $\exists h \in H^\infty$ $|h| \ge \delta > 0$ and

$\left| \arg \frac{|G|^2}{G^2} - \arg h \right| < \frac{\pi}{2} - \varepsilon$.

($\Rightarrow$)

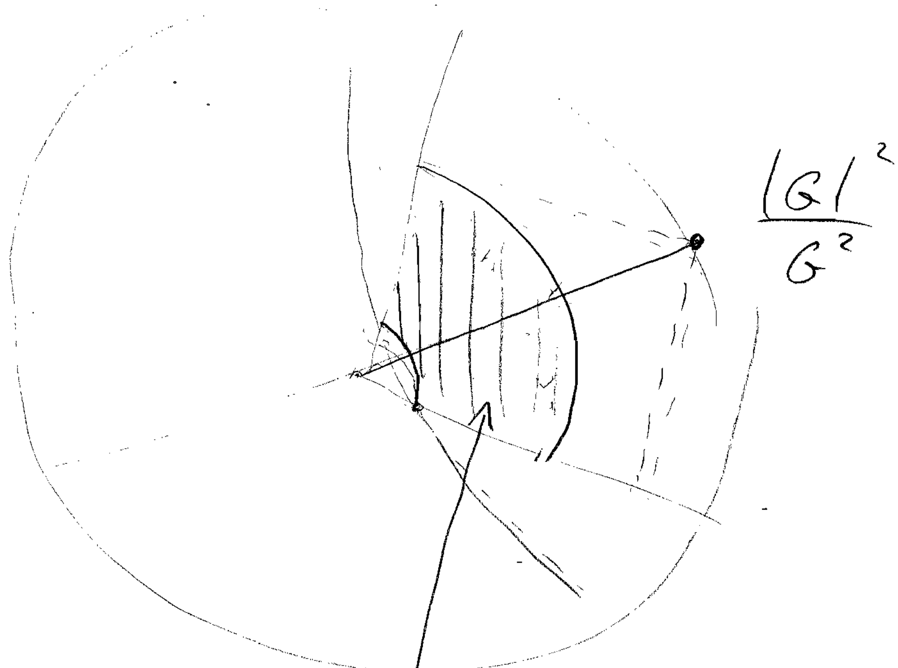


$(\Leftarrow)$



h is in this sector

(rescale) h



values of h should be in this sector, then $\left\| \frac{|G|^2}{G^2} - h \right\|_\infty < 1$