Homework assignment, Feb. 13, 2009.

Due Friday, Feb. 20.

1. Given a_k, $0 \le k \le n < \infty$ find

$$\min\left\{\|f\|_\infty : f \in H^\infty, f^{(k)}(0) = a_k, \quad \forall k,\ 0 \le k \le n\right\}$$

as the norm of a finite Hankel matrix.

2. Let H_φ be a Hankel operator such that $\|\varphi\|_\infty = \|H_\varphi\|$, and let $f \in H^2$, $f \ne 0$ be such that

$$\|H_\varphi f\| = \|H_\varphi\| \cdot \|f\|.$$

Prove that in this case

$$\varphi = \frac{H_\varphi f}{f} \qquad \text{a.e. on } \mathbb{T}.$$

3. Compute the reproducing kernel for the Bergman space A^2, consisting of all analytic in the unit disc $\mathbb{D}$ functions such that

$$\|f\|^2 := \iint_{\mathbb{D}} |f(z)|^2 dx dx < \infty$$

4. Let $z_k \in \mathbb{D}$, $k \ge 1$ $z_k \ne z_j$ for $j \ne k$. Define

$$\mathcal{K} := \operatorname{span}\{K_{z_j} : j \ge 1\};$$

here $K_{z_j}(z) = 1/(1 - \overline{z_j} z)$ are the reproducing kernels for H^2, and span always mean a *closed* linear span.

a) Show that if $\sum(1 - |z_k|) < \infty$, then $\mathcal{K} = K_B$, where $B = \prod_k b_{z_k}$ is the corresponding Blaschke product.

b) Show also that if $\sum(1 - |z_k|) < \infty$, then $\mathcal{K} = H^2$.

5. Let $h \in H^\infty$ and let $T_{\overline{h}}$ be the Toeplitz operator. Show that

$$T_{\overline{h}} K_\lambda = \overline{h(\lambda)} K_\lambda,$$

where K_λ is the reproducing kernel for H^2.