

Homework assignment, March 20, 2009.

Due Wednesday, April 1.

1. Show that the function $f(x) = \ln|x|$, $x \neq 0$ belongs to BMO and the function

$$f(x) = \begin{cases} \ln|x| & x > 0 \\ 0 & x \leq 0 \end{cases}$$

does not

2. # 1 on p. 261 of the text.

3. Show that if $f \in \text{BMO}$, then

$$|\nabla f(z)| \leq C \frac{\|f\|_*}{1 - |z|}, \quad z \in \mathbb{D}.$$

Here $f(z)$, $z \in \mathbb{D}$ denotes the harmonic extension of f from \mathbb{T} to the unit disc \mathbb{D} .

Hint: For $z = 0$ the inequality is trivial even for $f \in L^2$ (or even L^1). The general statement can be easily obtained using Garsia's norm by Möbius transformation of \mathbb{D} .

4. Let $\varphi \in C(\mathbb{D})$, such that

$$|f(z)| \leq \frac{C_1}{(1 - |z|)^2}$$

and such that the measure μ

$$d\mu = |\varphi(z)|(1 - |z|)dxdy$$

is Carleson, so

$$\iint_{\mathbb{D}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda}z|^2} d\mu(z) \leq C_2.$$

Show that in this case the Dirichlet problem

$$\Delta u(z) = \varphi(z), \quad u|_{\mathbb{T}} \equiv 0$$

has a bounded solution. Write estimate of the norm $\|u\|_{\infty}$ in terms of C_1 , C_2 .

Hint: Recall formula for the solution of the Dirichlet problem using the Green's formula. In the corresponding integral, the integral over a small hyperbolic neighborhood of λ is estimated using the first condition, and the integral over the rest using the Carleson measure condition.

This problem, together with the previous one gives you a direct proof of the fact that if the measure

$$|\nabla f(z)|^2(1 - |z|)dxdy$$

is Carleson, then Garsia's norm $\|f\|_G$ is finite.

5. Show that BMO is not separable.

Note, that BMO is much closer to L^∞ than to L^1 or L^2 . So, if you recall the proof of the fact that L^∞ is not separable, you can modify it so it works for BMO.