1. THE GROTHENDIECK RING

We work throughout over the complex numbers $\mathbb{C}$. Let **Var** (resp. **Sch**) be the category of varieties (resp. schemes) over $\mathbb{C}$, and let $K = K(\mathbf{Var})$ be the free abelian group on (isomorphism classes of) objects of **Var** modulo the relations

$$[X] = [Y] + [Y \setminus X]$$

whenever Y is a closed subvariety of X. K becomes a ring under the $\mathbb{Z}$-linear extension of the multiplication rule

$$[X][Y] := [X \times Y].$$

The additive identity in K is the class of the empty variety and the multiplicative unit is the class of a point $1_K = [\operatorname{Spec}\mathbb{C}]$. If we started with the category of all (finite type, say) schemes (over $\operatorname{Spec}\mathbb{C}$) and performed the same construction, we would arrive at the same ring. In particular, any (finite type) scheme X determines an element $[X]$ of K. We have $[X^{\mathrm{red}}] = [X]$ because X^{red} is a closed subscheme of X with trivial complement, so the class in the Grothendieck ring is not sensitive to nilpotents. The so-called "Lefschetz motive" $\mathbb{L} := [\mathbb{A}^1] \in K$ will arise frequently.

It is easy to prove by noetherian induction that for any Zariski locally trivial fibration

$$\begin{tikzcd}
F \arrow[r] & E \arrow[d] \\
 & B
\end{tikzcd}$$

we have $[E] = [F][B]$ in K. In particular, we have $[E] = [X]\mathbb{L}^r$ for any rank r vector bundle E over X.

Let $K_{\mathbb{L}}$ be the localization of K at the multiplicative set $\{\mathbb{L}, \mathbb{L}^2, \dots\}$ and define a decreasing filtration $F^\bullet$ on $K_{\mathbb{L}}$ by

$$F^m := \{\sum_i [X_i]\mathbb{L}^{-d_i} \in K_{\mathbb{L}} : \dim X_i - d_i \geq m \text{ for all } i\}.$$

Let $\overline{K}_{\mathbb{L}}$ be the completion of $K_{\mathbb{L}}$ with respect to $F^\bullet$, so an element of $\overline{K}_{\mathbb{L}}$ is represented by a Cauchy sequence (k_n) of elements of $K_{\mathbb{L}}$. Here a sequence is Cauchy if for any $m > 0$ there is a N such that $k_n - k_l \in F^m$ for all $l, n \geq N$.

It is not known whether the natural map to the completion $K_{\mathbb{L}} \to \overline{K}_{\mathbb{L}}$ is monic. In practice this is not a big problem, and only a curiosity.

2. MOTIVIC INTEGRATION

We work over the complex numbers $\mathbb{C}$. Let X be a variety of pure dimension d over $\mathbb{C}$. Let J^nX be the *scheme of n-jets* in X. It is defined by the existence of a natural bijection

$$\operatorname{Hom}_{\mathbf{Sch}/\mathbb{C}}(Y, J^nX) \cong \operatorname{Hom}_{\mathbf{Sch}/\mathbb{C}}(Y[t]/\langle t^{n+1}\rangle, X)$$

for every $\mathbb{C}$-scheme Y. We will often confuse J^nX with its set of $\mathbb{C}$-points

$$J^nX(\operatorname{Spec}\mathbb{C}) = \operatorname{Hom}_{\mathbf{Sch}/\mathbb{C}}(\operatorname{Spec}\mathbb{C}[t]/\langle t^{n+1}\rangle, X).$$

In particular, we write $\gamma \in J^nX$ as shorthand for $\gamma \in J^nX(\operatorname{Spec}\mathbb{C})$, and we understand that a "point" of J^nX means a $\mathbb{C}$-point of J^nX.

For $m > n$, let

$$\pi_n^m : J^m X \to J^n X$$

be the canonical projection. Each map π_n^m is an affine morphism. Let $J^\infty X = \varprojlim J^n X$ (the inverse limit is represented by a scheme by general theory because the transition morphisms in the inverse limit system are affine, but it is not of finite type). Evidently we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Sch}/\mathbb{C}}(Y, J^\infty X) &\cong \mathrm{Hom}_{\mathbf{Sch}/\mathbb{C}}(Y, \varprojlim J^n X) \\
&\cong \varprojlim \mathrm{Hom}_{\mathbf{Sch}/\mathbb{C}}(Y, J^n X) \\
&\cong \varprojlim \mathrm{Hom}_{\mathbf{Sch}/\mathbb{C}}(Y[t]/\langle t^{n+1}\rangle, X) \\
&\cong \mathrm{Hom}_{\mathbf{Ind\,Sch}/\mathbb{C}}(\mathrm{Spf}\, Y[[t]], X).
\end{aligned}$$

Let $\pi_n : J^\infty X \to J^n X$ be the canonical projection. We sometimes refer to points of $J^\infty X$ as *formal arcs* in X and points of $J^n X$ as *n-truncated arcs* in X.

A closed subscheme Y of X determines a function on formal arcs

$$\begin{aligned}
\mathrm{ord}_Y : J^\infty X &\to \mathbb{N} \cup \{\infty\} \\
\gamma &\mapsto \mathrm{ord}_Y \gamma
\end{aligned}$$

taking a formal arc γ to its *order of contact* with Y. More precisely, $\mathrm{ord}_Y \gamma$ is defined as follows. Let $U \cong \mathrm{Spec}\, A$ be an affine open neighborhood of the ($\mathbb{C}$) point $\pi_0 \gamma$ of X, so $Y \cap U$ is defined by an ideal I of A, and γ is a $\mathbb{C}$-algebra map $\gamma : A \to \mathbb{C}[[t]]$. For each $a \in A$, $\gamma(a)$ is a power series in t and we let $\mathrm{ord}_t \gamma(a)$ be the smallest power of t with a nonzero coefficient (or ∞ if $\gamma(a) = 0$). Then we define

$$\mathrm{ord}_Y \gamma := \min \{\mathrm{ord}_t \gamma(a) : a \in I\} \quad \in \mathbb{N} \cup \{\infty\}.$$

Note that $\mathrm{ord}_Y \gamma = \infty$ iff $\gamma \in J^\infty Y$. In other words, $\mathrm{ord}_Y \gamma$ is the minimum of the $\mathrm{ord}_Y \gamma(a)$ over all local sections a of the ideal sheaf of Y.

It follows from results of Denef and Loeser [DL99] that the sets $\pi_n(\mathrm{ord}_Y^{-1}(m))$ are constructible in $J^n X$ (hence they determine elements $[\pi_n(\mathrm{ord}_Y^{-1}(m))]$ of the Grothendieck ring K) and that the limit

$$\mu(\mathrm{ord}_Y^{-1}(m)) := \lim_{n \to \infty} \left([\pi_n(\mathrm{ord}_Y^{-1}(m))] \mathbb{L}^{-nd} \right)$$

exists in the completed Grothendieck ring $\overline{K}_{\mathbb{L}}$. In fact, they show that a much wider class of subsets of $J^\infty X$ is *measurable* in this sense. Their results are based on the model theory of formal power series rings and rely on quantifier elimination results of Pax, Kolchin, etc.

Remark 2.0.1. Their are differing conventions in the literature about the definition of μ, which some authors take to be

$$\mu(A) := \lim_{n \to \infty} \left([\pi_n(A)] \mathbb{L}^{-(n+1)d} \right).$$

Our convention is set up so that, for a smooth variety X, we have

$$\begin{aligned}\mu(J^\infty X) &= \lim_{n\to\infty}\left([\pi_n(J^\infty X)]\mathbb{L}^{-nd}\right)\\ &= \lim_{n\to\infty}\left([(J^n X)]\mathbb{L}^{-nd}\right)\\ &= \lim_{n\to\infty}\left([X]\mathbb{L}^{nd}\mathbb{L}^{-nd}\right)\\ &= [X].\end{aligned}$$

The *motivic integral* of ord_Y (or better: of $\mathbb{L}^{-\operatorname{ord}_Y}$) is defined by

$$\int_{J^\infty X}\mathbb{L}^{-\operatorname{ord}_Y}d\mu := \sum_{n\in\mathbb{N}}\mu(\operatorname{ord}_Y^{-1}(n))\mathbb{L}^{-n}.$$

Again, the fact that the infinite sum converges in $\overline{K}_{\mathbb{L}}$ is a result of [DL99]. In particular, the *motivic volume* of X is defined to be

$$\mu(X) := \int_{J^\infty X}\mathbb{L}^0 d\mu = \mu(J^\infty X) = \lim_{n\to\infty}\left([\pi_n(J^\infty X)]\mathbb{L}^{-dn}\right).$$

If X is smooth, then evidently the motivic volume of X is just $[X]$, but in general it is more difficult to compute.

The following theorems provide a means of computing motivic integrals:

Theorem 2.1. (Craw) *Let X be a smooth variety, $D=\sum_{i=1}^n a_iD_i$ an effective divisor on X with simple normal crossings. Then*

$$\int_{J^\infty X}\mathbb{L}^{-\operatorname{ord}_D} = \sum_{I\subseteq[n]}[D_I^\circ]\prod_{i\in I}\frac{\mathbb{L}-1}{\mathbb{L}^{a_i+1}-1},$$

where the sum is over all subsets I of $[n]=\{0,\dots,n\}$, and

$$\begin{aligned}D_I &:= \cap_{i\in I}D_i\\ D_I^\circ &:= D_I\setminus\cup_{j\in[n]\setminus I}D_j,\end{aligned}$$

with the convention $D_\emptyset = X$.

For a map $f: X\to Y$ between schemes of the same pure dimension n, with X smooth, we define an ideal sheaf $\mathscr{I}^\Delta_{X/Y}$, called the *discrepancy ideal* as follows. Locally, take a local nowhere vanishing n-form η on X and consider the ideal generated by all expressions $df(\alpha)/\eta$ where α is an n-form on Y and

$$df : \wedge^n f^*\Omega_Y \to \wedge^n\Omega_X = \omega_X$$

is the n^{th} exterior power of the differential of f. Since two nowhere vanishing n-forms differ by an invertible function on, the ideal does not depend on the choice of trivialization, so the locally defined ideals glue. Let $\Delta_{X/Y}$ denote the closed subscheme of X with ideal sheaf $\mathscr{I}^\Delta_{X/Y}$.

Theorem 2.2. (Denef-Loeser) *Let $f: X\to Y$ be a birational morphism of varieties, with X smooth of dimension d. Let Z be a closed subscheme of Y and let $f^{-1}Z$ be the closed subscheme of X defined by the inverse image ideal sheaf of Z. Then*

$$\int_{J^\infty Y}\mathbb{L}^{-\operatorname{ord}_Z}d\mu = \int_{J^\infty Y}\mathbb{L}^{-\operatorname{ord}_{\Delta_{X/Y}}-\operatorname{ord}_Z}d\mu.$$

It is often convenient, for computational purposes, to stratify J^nX as

$$J^nX = J^n_0X \coprod J^n_1X \coprod \cdots \coprod J^n_{n+1}X,$$

where

$$J^n_mX = \{\gamma \in J^nX : \operatorname{ord}_{X^{\mathrm{sing}}} \gamma = m\}.$$

Technically speaking, we haven't defined the order of contact of a truncated arc γ with a closed subscheme Y, but it is just as for formal arcs, except there is no notion of infinite order contact; the highest possible contact order for an n-jet is $(n+1)$, and this occurs iff the n-jet factors through Y (is in the image of $J^nY \to J^nX$). Evidently we could continue stratifying J^nX by stratifying $J^n_{n+1}X = J^nX^{\mathrm{sing}}$ in the same way. We will use this stratification in the next section.

Note $J^n_0X = (\pi^n_0)^{-1}(X^{\mathrm{sm}})$ is an open subset of J^nX. It is nothing but the space of n-jets into the smooth locus of X and is hence an $\mathbb{A}^{dn}$ bundle over X^{sm} when X is of pure dimension d.

3. Example: The Node and Cusp

In this section, we compute the motivic volume of the node $N := \operatorname{Spec}\mathbb{C}[x,y]/\langle xy\rangle$ and the cusp $C := \operatorname{Spec}\mathbb{C}[x,y]/\langle y^2 - x^3\rangle$ straight from the definitions. We accomplish this by noticing a certain "periodicity" for the jet spaces—it turns out that the space of formal arcs into the node or cusp with a specified order of contact with the singular locus can be identified with the space of formal arcs into the smooth locus. Since the latter spaces are easy to understand, we can compute the motivic volume by using the stratification of formal arc spaces by order of contact with the singular locus. I suspect this technique generalizes to compute other motivic integrals, but I have not attempted to lay out any general setup.

Throughout this section, we write Π_n for the image of the truncation π_n.

3.1. The node.

A formal arc in the node N is a ring homomorphism

$$\gamma : \mathbb{C}[x,y]/\langle xy\rangle \to \mathbb{C}[[t]].$$

That is, γ is a ring homomorphism

$$\begin{array}{rcl} \gamma : \mathbb{C}[x,y] & \to & \mathbb{C}[[t]] \\ x & \mapsto & a_0 + a_1t + a_2t^2 + \ldots \\ y & \mapsto & b_0 + b_1t + b_2t^2 + \ldots \end{array}$$

where the coefficients $a_0, b_0, a_1, b_1, \dots$ satisfy the relations

$$\begin{aligned} R_0 &= a_0b_0 \\ R_1 &= a_0b_1 + a_1b_0 \\ R_2 &= a_0b_2 + a_1b_1 + a_2b_0 \\ R_3 &= a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0 \\ &\vdots \\ R_m &= \sum_{i=0}^{m} a_ib_{m-i} \\ &\vdots \end{aligned}$$

Using this choice of coordinates, we make the identifications

$$\begin{aligned} J^n\mathbb{A}^2_{x,y} &\cong \operatorname{Spec}\mathbb{C}[a_0, b_0, \dots, a_n, b_n] \\ J^\infty\mathbb{A}^2_{x,y} &\cong \operatorname{Spec}\mathbb{C}[a_0, b_0, a_1, b_1, \dots] \\ J^nN &\cong \operatorname{Spec}\mathbb{C}[a_0, b_0, \dots, a_n, b_n]/\langle R_0, \dots, R_n\rangle \\ J^\infty N &\cong \operatorname{Spec}\mathbb{C}[a_0, b_0, a_1, b_1, \dots]/\langle R_0, R_1, \dots\rangle. \end{aligned}$$

The singular locus of N is given by the ideal $\langle x, y\rangle$, so if we let

$$I_k := \langle a_0, b_0, \dots, a_k, b_k\rangle,$$

then in the stratification of $J^\infty N$ in terms of order of contact with the singular locus, $J^\infty_{\geq m}N$ is the closed subset of $J^\infty N$ determined by I_{m-1}, and $J^\infty_m N$ is the open subset of $J^\infty_{\geq m}N$ where at least one of a_m, b_m is nonzero. That is,

$$J^\infty_m N \cong \operatorname{Spec}\mathbb{C}[a_0, b_0, \dots]_{\langle a_m, b_m\rangle}/\langle a_0, b_0, \dots, a_{m-1}, b_{m-1}, R_0, R_1, \dots\rangle.$$

Note $J^0_0N = N^{\text{sm}}$ and $J^0_1 = N^{\text{sing}}$. We have $[N^{\text{sm}}] = 2(\mathbb{L} - 1)$.

The relations $R_0, \dots, R_{2m-1}$ are clearly trivial when restricted to $J^\infty_m N$, while the relations $R_{2m}, R_{2m+1}, \dots$ reduce to

$$\begin{aligned} R_{2m} &= a_mb_m \\ R_{2m+1} &= a_mb_{m+1} + a_{m+1}b_m \\ R_{2m+2} &= a_mb_{m+2} + a_{m+1}b_{m+1} + a_{m+2}b_m \\ &\vdots \end{aligned}$$

This proves that the "shift operator"

$$\begin{aligned} s : \operatorname{Spec}\mathbb{C}[a_0, b_0, \dots] &\to \operatorname{Spec}\mathbb{C}[a_0, b_0, \dots] \\ (a_0, a_1, \dots) &\mapsto (a_m, a_{m+1}, \dots) \\ (b_0, b_1, \dots) &\mapsto (b_m, b_{m+1}, \dots) \end{aligned}$$

descends to an isomorphism $s : J^\infty_m N \to J^\infty_0 N$ whose inverse is given by

$$\begin{aligned} (a_0, a_1, \dots) &\mapsto (\underbrace{0, \dots, 0}_{m}, a_0, a_1, \dots) \\ (b_0, b_1, \dots) &\mapsto (\underbrace{0, \dots, 0}_{m}, b_0, b_1, \dots). \end{aligned}$$

This shift operator isomorphism is compatible with the projections in the sense that there are commutative diagrams

$$\begin{tikzcd}
J_m^\infty N \arrow[r, "s"] \arrow[d, "\pi_{m+k}"'] & J_0^\infty N \arrow[d, "\pi_k"] \\
\Pi_{m+k} \cap J_m^{m+k} N \arrow[r, "\cong"] & \Pi_k \cap J_0^k N
\end{tikzcd}$$

for any $k \in \mathbb{N}$, where the bottom isomorphism is given by the truncated shift operator

$$\begin{array}{rcl}
(0, \ldots, 0, a_m, \ldots, a_{m+k}) & \mapsto & (a_m, \ldots, a_{m+k}) \\
(0, \ldots, 0, b_m, \ldots, b_{m+k}) & \mapsto & (b_m, \ldots, b_{m+k}).
\end{array}$$

Since $J_0^n N$ is just the space of jets into the smooth part of the node, it is an $\mathbb{A}^n$ fibration over the node, so $[J_0^n N] = 2\mathbb{L}^n(\mathbb{L}-1)$.

For any $n \in \mathbb{N}$, using the truncated shift isomorphisms with

$$(m, k) = (0, n), (1, n-1), \ldots, (n, 0),$$

we find

$$\begin{array}{rcl}
[\Pi_n \cap J_0^n N] & = & 2\mathbb{L}^n(\mathbb{L}-1) \\
[\Pi_n \cap J_1^n N] & = & 2\mathbb{L}^{n-1}(\mathbb{L}-1) \\
 & \vdots & \\
[\Pi_n \cap J_{n-1}^n N] & = & 2\mathbb{L}(\mathbb{L}-1) \\
[\Pi_n \cap J_n^n N] & = & 2(\mathbb{L}-1).
\end{array}$$

Certainly $J_{n+1}^n N$ consists of a single point corresponding to the n-jet where all the a_i and b_i are zero. Evidently this truncated arc lifts to a formal arc by setting all the a_i and b_i to zero, so we also have $[\Pi_n \cap J_{n+1}^n] = 1$.

Adding up the contributions from the strata, we find

$$\begin{array}{rcl}
[\Pi_n] & = & [\Pi_n \cap J_0^n N] + \cdots + [\Pi_n \cap J_{n+1}^n N] \\
 & = & 2(\mathbb{L}^n + \mathbb{L}^{n-1} + \cdots + 1)(\mathbb{L}-1) + 1,
\end{array}$$

so

$$[\Pi_n]\mathbb{L}^{-n} = 2(1 + \mathbb{L}^{-1} + \cdots + \mathbb{L}^{-n})(\mathbb{L}-1) + \mathbb{L}^{-n},$$

and hence

$$\mu(N) = \lim_{n\to\infty} \left([\Pi_n]\mathbb{L}^{-n}\right) = 2\frac{(\mathbb{L}-1)}{(1-\mathbb{L}^{-1})} = 2\mathbb{L}.$$

This is easily reconciled with the change of variables formula. Indeed, we can resolve N be its normalization $f : \mathbb{A}_x^1 \coprod \mathbb{A}_y^1 \to N$. The differential of f gives a surjection $f^*\Omega_N \to \omega_{\mathbb{A}^1 \coprod \mathbb{A}^1}$ since it takes dx to dx and dy to dy. Therefore, the discrepancy ideal is trivial, and the change of variables formula reads

$$\begin{aligned}\mu(N) &= \int_{J^\infty N} \mathbb{L}^0 d\mu \\ &= \int_{J^\infty(\mathbb{A}^1 \coprod \mathbb{A}^1)} \mathbb{L}^0 d\mu \\ &= \mu(\mathbb{A}^1 \coprod \mathbb{A}^1) \\ &= 2\mathbb{L}.\end{aligned}$$

3.2. **The cusp.** A formal arc in the cusp $C = \operatorname{Spec} \mathbb{C}[x,y]/\langle x^3 = y^2\rangle$ is a ring homomorphism

$$\gamma : \mathbb{C}[x,y]/\langle x^3 = y^2\rangle \to \mathbb{C}[[t]],$$

which we view as a ring homomorphism

$$\begin{aligned}\gamma : \mathbb{C}[x,y] &\to \mathbb{C}[[t]] \\ x &\mapsto a_0 + a_1 t + a_2 t^2 + \dots \\ y &\mapsto b_0 + b_1 t + b_2 t^2 + \dots\end{aligned}$$

where the coefficients $a_0, b_0, a_1, b_1, \dots$ satisfy the relations

$$\begin{array}{lrcl} R_0 : & a_0^3 & = & b_0^2 \\ R_1 : & 3a_0^2 a_1 & = & 2b_0 b_1 \\ R_2 : & 3a_0 a_1^2 + 3a_0^2 a_2 & = & 2b_0 b_2 + b_1^2 \\ R_3 : & 3a_0^2 a_3 + 6a_0 a_1 a_2 + a_1^3 & = & 2b_1 b_2 \\ & & \vdots & \\ R_m : & \displaystyle\sum_{\substack{(i,j,k)\in\mathbb{N}^3 \\ i+j+k=m}} a_i a_j a_k & = & \displaystyle\sum_{\substack{(i,j)\in\mathbb{N}^2 \\ i+j=m}} b_i b_j \\ & & \vdots & \end{array}$$

We have $J^\infty C \cong \operatorname{Spec} \mathbb{C}[a_0, b_0, a_1, b_1, \dots]/\langle R_0, R_1, \dots\rangle$. As with the node, the singular locus of the cusp is given by $\langle x, y\rangle$, so $J^\infty_m C$ is the locus of formal arcs where $a_0 = b_0 = \dots = a_{m-1} = b_{m-1} = 0$ but at least one of a_m, b_m is non-zero.

First I claim that for any $m \in \mathbb{N}$, the scheme $J^\infty_{2m+1} C$ is empty. That is, there are no formal arcs in the cusp having odd contact order with its singular locus. Note that $J^\infty_{2m+1} C$ is an open subscheme of the closed subscheme of $J^\infty C$ cut out by the ideal

$$I = \langle a_0, \dots, a_{2m}, b_0, \dots, b_{2m}\rangle.$$

This open subscheme is the one where at least one of a_{2m+1}, b_{2m+1} is nonzero, so to prove the claim, it suffices to prove the containment of ideals

$$\langle a_{2m+1}, b_{2m+1}\rangle \subseteq \sqrt{I + \langle R_0, R_1, \dots\rangle}. \tag{1}$$

Modulo I, we have

$$\begin{array}{lrcl} R_{4m+2}: & 0 & = & b_{2m+1}^2 \\ R_{4m+4}: & 0 & = & b_{2m+2}^2+\dots \\ R_{4m+6}: & 0 & = & b_{2m+3}^2+\dots \\ & & \vdots & \\ R_{6m+2}: & 0 & = & b_{3m+1}^2+\dots \\ R_{6m+3} & a_{2m+1}^3 & = & 2b_{3m+1}b_{3m+2}+\dots, \end{array}$$

where the $\dots$ are terms in the radical ideal generated by the previous relations (e.g. the $\dots$ in R_{4m+4} stand for $2b_{2m+1}b_{2m+3}$, but b_{2m+1} is in the radical ideal generated by R_{4m+2}), so the resulting containment of ideals (1) is proved. Note that the odd indexed relations $R_{4m+3},\dots,R_{6m+1}$ are not needed (they are already contained in the radical ideal generated by the other relations and I).

Next I claim that the map

$$\begin{array}{rcl} s:\operatorname{Spec}\mathbb{C}[a_0,a_1,\dots,b_0,b_1,\dots] & \to & \operatorname{Spec}\mathbb{C}[a_0,a_1,\dots,b_0,b_1,\dots] \\ (a_0,a_1,\dots) & \mapsto & (a_{2m},a_{2m+1},\dots) \\ (b_0,b_1,\dots) & \mapsto & (b_{3m},b_{3m+1},\dots) \end{array}$$

descends to an isomorphism $(J_{2m}^\infty C)^{\mathrm{red}}\to(J_0^\infty C)^{\mathrm{red}}$. We have no quams about passing to reduced schemes, since we only care about images in the Grothendieck ring. The inverse of this morphism will be given by

$$\begin{array}{rcl} \operatorname{Spec}\mathbb{C}[a_0,a_1,\dots,b_0,b_1,\dots] & \to & \operatorname{Spec}\mathbb{C}[a_0,a_1,\dots,b_0,b_1,\dots] \\ (a_0,a_1,\dots) & \mapsto & (\underbrace{0,\dots,0}_{2m},a_0,a_1,\dots) \\ (b_0,b_1,\dots) & \mapsto & (\underbrace{0,\dots,0}_{3m},b_0,b_1,\dots). \end{array}$$

The radical ideal

$$\sqrt{\langle a_0,\dots,a_{2m-1},b_0,\dots,b_{2m-1},R_0,R_1,\dots\rangle}$$

defines a closed subset of $\operatorname{Spec}\mathbb{C}[a_0,\dots,b_0,\dots]$, and $(J_{2m}^\infty)^{\mathrm{red}}$ is the open set of this closed subset where at least one of a_{2m},b_{2m} is nonzero. First of all, we can prove that $b_0,\dots,b_{3m-1}$ are in this radical ideal just as we did in the previous paragraph. Modulo the ideal

$$\langle a_0,\dots,a_{2m-1},b_0,\dots,b_{3m-1}\rangle,$$

the relations $R_0,\dots,R_{6m-1}$ are trivial, and the other relations can be written

$$\begin{array}{lrcl} R_{6m}: & a_{2m}^3 & = & b_{3m}^2 \\ R_{6m+1}: & 3a_{2m}a_{2m+1} & = & 2b_{3m}b_{3m+1} \\ R_{6m+2}: & 3a_{2m}^2a_{2m+2}+3a_{2m+1}^2a_{2m+2} & = & 2b_{3m}b_{3m+2}+b_{3m+1}^2 \\ & & \vdots & \end{array}$$

so we see that our map s exchanges R_{6m+k} and R_k. This proves the claim.

We have $[\Pi_n\cap J_0^nC]=\mathbb{L}^n(\mathbb{L}-1)$ because $J_0^nC=\Pi_n\cap J_0^nC$ is an $\mathbb{A}^n$-bundle over $C^{\mathrm{sm}}\cong\mathbb{A}^1\setminus\{0\}$. As with the node, we get isomorphisms $\Pi_{2m+k}\cap J_{2m}^{2m+k}C\cong\Pi_0\cap J_0^kC$ by considering a truncated version of s. For $n\in\mathbb{N}$ even, we apply these isomorphisms with

$$(m,k)=(0,n),(1,n-1),\dots,(n/2,0),$$

and use the previous equality to find

$$\begin{array}{rcl}
[\Pi_n \cap J_0^n C] & = & \mathbb{L}^n(\mathbb{L}-1) \\
[\Pi_n \cap J_2^n C] & = & \mathbb{L}^{n-2}(\mathbb{L}-1) \\
 & \vdots & \\
[\Pi_n \cap J_n^n C] & = & (\mathbb{L}-1).
\end{array}$$

(There is a similar expression for n odd.) As with the node, $J_{n+1}^n C = \Pi_n \cap J_{n+1}^n C$ consists of a single point, so we compute

$$\begin{array}{rcl}
[\Pi_n] & = & [\Pi_n \cap J_0^n C] \coprod [\Pi_n \cap J_2^n C] \cdots [\Pi_n \cap J_n^n C] \coprod [\Pi_n \cap J_{n+1}^n C] \\
 & = & (\mathbb{L}^n + \mathbb{L}^{n-2} + \cdots + 1)(\mathbb{L}-1) + 1.
\end{array}$$

From this, we obtain the motivic volume of the cusp:

$$\begin{array}{rcl}
\mu(C) & = & \lim\limits_{n\to\infty} \big([\Pi_n]\mathbb{L}^{-n}\big) \\
 & = & \lim\limits_{n\to\infty} \big((1 + \mathbb{L}^{-2} + \mathbb{L}^{-4} + \cdots + \mathbb{L}^{-n})(\mathbb{L}-1)\big) \\
 & = & \dfrac{\mathbb{L}-1}{1-\mathbb{L}^{-2}}
\end{array}$$

We could also compute this using the change of variables formula and Craw's formula as follows. We may resolve the cusp by its normalization, which is Spec of the ring map

$$\begin{array}{rcl}
\mathbb{C}[x,y]/\langle x^3 = y^2\rangle & \to & \mathbb{C}[z] \\
x & \mapsto & z^2 \\
y & \mapsto & z^3.
\end{array}$$

The differential of this map takes dx to $2zdz$ and y to $3z^2dz$. Since dz trivializes $\omega_{\mathbb{A}_z^1}$, the discrepancy ideal is $\Delta_{\mathbb{A}^1/C} = \langle z\rangle$, and the corresponding closed subscheme is the origin of $\mathbb{A}_z^1$. By the change of variables formula, and Craw's formula, we compute

$$\begin{array}{rcl}
\mu(C) & = & \displaystyle\int_{J^\infty C} \mathbb{L}^0 d\mu \\
 & = & \displaystyle\int_{J^\infty \mathbb{A}_z^1} \mathbb{L}^{-\operatorname{ord}_z} d\mu \\
 & = & [D_\emptyset^\circ] + [D_1^\circ]\dfrac{\mathbb{L}-1}{\mathbb{L}^2-1} \\
 & = & (\mathbb{L}-1) + \dfrac{\mathbb{L}-1}{\mathbb{L}^2-1} \\
 & = & \dfrac{\mathbb{L}^2(\mathbb{L}-1)}{\mathbb{L}^2-1} \\
 & = & \dfrac{\mathbb{L}-1}{1-\mathbb{L}^2}
\end{array}$$

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