August 2, 2008

Hi Jonathan,

Hope all is well with you. Are you still in Providence by any chance? I'll be up there this weekend looking for an apartment. Let me know if you're around. [...] I saw your paper *A hyperelliptic Hodge integral* on the `arXiv`. In trying to prove your result by another method, I established a similar formula:

$$\int_{\overline{\mathcal{M}}_{0;2g+2}(B\mathbb{Z}_2)} \frac{c(\mathbb{E}^\vee)^2}{1-\psi} \quad = \quad (-1)^{g-1}\frac{(g-1)}{4}. \tag{1}$$

which seemed to disagree with your result. [...] Eventually I discovered that the disparity arises because your ψ class is from the *covering curve*, which makes it half of the usual ψ class. [...] In more standard notation, your formula should read:

$$\int_{\overline{\mathcal{M}}_{0;2g+2}(B\mathbb{Z}_2)} \frac{c(\mathbb{E}^\vee)^2}{1-\frac{1}{2}\psi} \quad = \quad \left(\frac{-1}{4}\right)^g \tag{2}$$

Let me explain my approach. My notation for GW invariants is: $I = (i_1, \dots, i_m)$ is a sequence of *insertions* (elements of a basis for the cohomology of the target assumed homogeneous with respect to the splitting of the target into connected components, which is especially relevant in the orbifold theory where the "target" is the inertia stack), $a : I \to \mathbb{Z}_{\geq 0}$ is a function keeping track of descendants, $\overline{\mathcal{M}}_{0;I}(\mathscr{X}, d)$ is the component of $\overline{\mathcal{M}}_{0;m}(\mathscr{X}, d)$ where the evaluation maps take values in the correct component of the target, and I write

$$\langle I^a \rangle_d^{\mathscr{X}} = \int_{[\overline{\mathcal{M}}_{0;I}(\mathscr{X},d)]^{vir}} \psi_I^a$$

for the GW invariant, with the understanding $\psi_I^a = \prod_{j=1}^m e_j^*(i_j)\psi_j^{a(i_j)}$. (The ψ classes are from the coarse curve, as usual.) The efficacy of such a notation derives from the fact that so many gluing/degeneration/boundary relations among GW invariants are written in the form

$$\langle I^a, \dots \rangle \quad = \sum_{K \coprod L = I} \langle K^{a|K}, \dots \rangle \langle L^{a|L}, \dots \rangle,$$

as we will see in a second.

In the case $\mathscr{X} = [\mathbb{C}^2/\mathbb{Z}_2]$, there is only $d = 0$, so we'll drop it from the notation, and the target has no interesting cohomology classes so I is just a $0,1$-sequence (0 for the fundamental class of $\mathscr{X}$ and 1 for the fundamental class of the twisted sector). The GW invariants are T-equivariant with the 2-torus T acting on $\mathscr{X}$ through the inclusion $\mathbb{C}^* \times \mathbb{C}^* \hookrightarrow \mathbb{C}^2$, and hence on the two dual Hodge bundle factors by scaling:

$$\begin{aligned} \langle I^a \rangle := \langle I^a \rangle_0^{[\mathbb{C}^2/\mathbb{Z}_2]} \quad &= \quad \int_{\overline{\mathcal{M}}_{0;I}(B\mathbb{Z}_2)} \psi_I^a e_T(\mathbb{E}^\vee \oplus \mathbb{E}^\vee) \\ &= \quad \sum_{i,j} (-1)^{i+j} t_1^i t_2^j \int_{\overline{\mathcal{M}}_{0;I}(B\mathbb{Z}_2)} \lambda_{g-i}\lambda_{g-j}\psi_I^a \qquad \in \mathbb{Q}[t_1, t_2] \end{aligned}$$

(We assume throughout that I has at least one "1" so the stable map factors through $[0/\mathbb{Z}_2]$.) You only need to consider the "evaluation" at $t_1 = t_2 = 1$, because the total

Chern class $c(\mathbb{E}^\vee)$ is nothing but the equivariant Euler class $e_T(\mathbb{E}^\vee)$ evaluated at $t_i = 1$. This generality helps clarify the situation.

Recall (from Chapter 10 of [Cox/Katz], say) the *topological recursion relations* (TRRs): For any I, a as above, any insertions i, j, k, and any $b, c, d \in \mathbb{Z}_{\geq 0}$, there is a stabilization map $\overline{\mathcal{M}}_{0;I,i,j,k}(B\mathbb{Z}_2) \to \overline{M}_{0;3}$ obtained by retaining only the coarse curve and the markings carrying the last three insertions. The cotangent line bundle $\mathbb{L}_1$ on the point $\overline{M}_{0;3}$ is trivial, but the stabilization map gives an isomorphism on cotangent lines away from the locus where the component of C containing the i marking is contracted. This is exactly the boundary divisor $D(i|j,k)$ where the curve C splits into (at least) two components with the i marking on one and the markings carrying j, k on the other. Up to finite maps, this divisor is expressed as a product of stable map spaces indexed by splittings of the remaining markings:

$$\coprod_{K \coprod L = I} \coprod_{h=0,1} \overline{\mathcal{M}}_{0;K,i,h}(B\mathbb{Z}_2) \times \overline{\mathcal{M}}_{0;L,j,k,h}(B\mathbb{Z}_2) \to D(i|j,k).$$

Integrating the cohomology class $\psi_I^a \psi_i^b \psi_j^c \psi_k^d e(\mathbb{E}^\vee \oplus \mathbb{E}^\vee)$ against both sides of the equality $\psi_i = D(i|j,k)$ and using this description of the boundary, we obtain

$$\begin{aligned} \langle I^a, \tau_{b+1} i, \tau_c j, \tau_d k \rangle &= 2 \sum_{K \coprod L = I} \langle K^{a|K}, \tau_b i, 1 \rangle \langle L^{a|L}, \tau_c j, \tau_d k, 1 \rangle \\ &\quad + 2 t_1 t_2 \sum_{K \coprod L = I} \langle K_{a|K}, \tau_b i, 0 \rangle \langle L_{a|L}, \tau_c j, \tau_d k, 0 \rangle. \end{aligned} \tag{3}$$

The factors of 2 arise because of the extra automorphisms in the product; the $t_1 t_2$ term arises because the dual hodge bundle $\mathbb{E}^\vee$ contains a trivial factor on the divisor of nodal domains obtained by gluing at non-stacky markings. Obviously this relation determines all descendant GW invariants of $[\mathbb{C}^2/\mathbb{Z}_2]$ in terms of the (known) non-descendant invariants, so any theorem about such invariants is basically a matter of formula mongering.

I certainly don't discourage this formula mongering: Renzo and I have established various nice formulas for invariants of $[\mathbb{C}^2/\mathbb{Z}_2]$ in this manner. There is a similar story for $[\mathbb{C}^2/\mathbb{Z}_n]$: using only the TRRs and a trick of Renzo's it is possible to derive recursions (amenable to exact solution) for all (genus zero) descendant invariants of the latter. I kind of lost interest in this because I got so sick of the (very difficult) formula mongering. I think there is a meta theorem to the effect that the TRRs plus any meaningful geometric relation will determine all the GW invariants of $[\mathbb{C}^2/\mathbb{Z}_n]$. Probably WDVV on the degree $1/n$ invariants of $\mathbb{P}(1,1,n)$ will yield such a "geometric relation" for any n.

Anyway, after setting up notation, proving your theorem is easy. We will only need (3) in the case $I = (1^{\otimes 2g-1})$, $j = k = 1$, $a = 0$, $c = d = 0$. We set $t_1 = t_2 = 1$ from now on.

As long as $g > 1$, we get:

$$\begin{aligned}(4)\qquad \langle 1^{\otimes 2g-1}, \tau_{b+1}1, 1, 1\rangle &= 2\sum_i \big(2g-1\big)\,\langle 1^{\otimes 2i}, \tau_b 1, 1\rangle\langle 1^{\otimes 2(g-i)-1}, 1, 1, 1\rangle \\ &\quad +2\sum_i \binom{2g-1}{2i+1}\,\langle 1^{\otimes 2i+1}, \tau_b 1, 0\rangle\langle 1^{\otimes 2(g-i)-2}, 1, 1, 0\rangle \\ &= 2\sum_i \binom{2g-1}{2i}\,\langle 1^{\otimes 2i+1}, \tau_b 1\rangle\langle 1^{\otimes 2(g-i)+2}\rangle \\ &\quad +\langle 1^{\otimes 2g-1}, \tau_{b-1}1\rangle\end{aligned}$$

(For the second equality, note that the second term in the second sum vanishes because of the trivial insertion, except when $i = g-1$, in which case it is $1/2$. Then use the String Equation to remove the trivial insertion in the first term of the second sum, which we can do since $g > 1$.) Obviously this recursion determines the invariants we are interested in. Multiplying by 2^{-b-1}, summing over $b = 0, 1, \ldots$ and moving the last term to the LHS, we get:

$$\langle 1^{\otimes 2g+1}, \frac{1}{1-\frac{1}{2}\psi}\rangle - \frac{1}{4}\langle 1^{\otimes 2g-1}, \frac{1}{1-\frac{1}{2}\psi}\rangle = \sum_i \binom{2g-1}{2i}\langle 1^{\otimes 2i+1}, \frac{1}{1-\frac{1}{2}\psi}\rangle\langle 1^{\otimes 2(g-i)+2}\rangle$$

This is not strictly accurate because the $b = 0$ term $\langle 1^{\otimes 2g+1}, 1\rangle$ is missing from the LHS. However, this will be exactly offset by *defining* the unstable term

$$\langle 1, \frac{1}{1-\frac{1}{2}\psi}\rangle := 1,$$

which has the effect of adding this term to the RHS.

Now we put this in generating function form. Define:

$$\begin{aligned} h &:= x + \sum_{g\geq 1}\langle 1^{\otimes 2g+1}, \frac{1}{1-\frac{1}{2}\psi}\rangle \frac{x^{2g+1}}{(2g+1)!} \\ F &:= \sum_{g\geq 1}\langle 1^{\otimes 2g+2}\rangle \frac{x^{2g-1}}{(2g-1)!}\end{aligned}$$

The fact that

$$\begin{aligned} F &= \sum_{g\geq 1}\left(-(t_1+t_2)\int_{\overline{\mathcal{M}}_{0;2g+2}(B\mathbb{Z}_2)} \lambda_g\lambda_{g-1}\right)\Bigg|_{t_1=t_2=1} \frac{x^{2g-1}}{(2g-1)!} \\ &= -\tan(x/2)\end{aligned}$$

is well-known (see the original Bryan/Graber CRC paper, for example). Formula (2) is equivalent to saying

$$h = x + \sum_{g\geq 1}\left(\frac{-1}{4}\right)^g \frac{x^{2g+1}}{(2g+1)!} = 2\sin(x/2).$$

The relation above is equivalent to the equality of the coefficients of x^{2g-1} in the equation

$$h'' - \frac{1}{4}h = h'F$$

when $g > 1$. One can check equality of low order terms ($g = 0, 1$) by hand, so we conclude

$$h'' - \frac{1}{4}h \quad = \quad h'(-\tan(x/2)). \tag{5}$$

Now (2) follows by checking that $h = 2\sin(x/2)$ is a solution to (5), which is a simple calculus exercise I leave to you.

To establish my formula (1), define:

$$f \quad := \quad \frac{1}{2}x + \sum_{g\geq 1}\langle 1^{\otimes 2g+1}, \frac{1}{1-\psi}\rangle \frac{x^{2g+1}}{(2g+1)!}$$

Then summing over $b = 0, 1, \ldots$ in (4) without putting in the factors of 2, we get a relation implying

$$f'' - f \quad = \quad 2f'F. \tag{6}$$

You can check with Mathematica that the coefficients of the series f are given as in (1).

—Danny