Probability: Final Exam Solutions

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Part I.

This material is in the text or your notes.

(1) **(3 Points)** Give the probability $P(X = k)$ when X is a discrete random variable with a....
 (a) binomial distribution based on n trials with success probability p.
 (b) Poisson distribution with expected value λ.
 (c) negative binomial distribution counting the number of success probability p trials up to the n^{th} success.

(2) **(3 Points)** Give the definitions of the beta and gamma functions, explain how they are related, and give the functional equation for the gamma function.

(3) **(4 Points)** Give the density functions for the gamma distribution and the beta distribution (with parameters α, β) and give the expected value of each distribution. Give the cumulative distribution function for the exponential distribution.

Solution. Many of you seem to have forgotten the CDF:

$$\begin{aligned} F(x) &= \int_0^x \frac{e^{-y/\beta}}{\beta} dy \\ &= \left[-e^{-y/\beta} \right]_0^x \\ &= 1 - e^{-x/\beta}. \end{aligned}$$

(4) **(3 Points)** Give Stirling's approximation of $n!$ and name two places where we used it.

Solution. I was hoping you would mention the de Moivre-Laplace Theorem (the Central Limit Theorem for Bernoulli trials) and Pólya's theorem on random walks.

(5) **(3 Points)** Give the definition of the n^{th} harmonic number H_n and say something intelligent about the harmonic numbers.

Solution. $H_n = 1 + 1/2 + \cdots + 1/n$. I was hoping you would mention the recursion from your homework, or the generating function

$$\sum_n H_n x^n = -\frac{\ln(1-x)}{1-x},$$

or perhaps the coupon collecting problem or the Markov chain problem whose solutions involved the harmonic numbers.

(6) **(3 Points)** Give Chebyshev's two bounds and state the Weak Law of Large Numbers.

(7) **(2 Points)** State the Central Limit Theorem.

Part II.

(8) **(5 Points)** Suppose X, Y are independent continuous random variables with density functions

$$\begin{aligned} f(x) &= \begin{cases} x/2, & 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases} \\ g(x) &= \begin{cases} 3x^2, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

(respectively). Calculate $E(X)$ and $E(Y)$. Calculate the density function for $X+Y$ and use it to calculate $E(X+Y)$.

Solution. $E(X) = \int_0^2 x(x/2) = [x^3/6]_0^2 = 4/3$ and $E(Y) = \int_0^1 3x^3 = [3x^4/4]_0^1 = 3/4$. The density $h(x)$ for $X+Y$ is given by convolution:

$$h(x) = \int_{\mathbb{R}} f(y)g(x-y)dy.$$

The integrand is zero unless $0 \leq y \leq 2$ and $0 \leq x - y \leq 1$ (i.e. unless $0 \leq y \leq 2$ and $x - 1 \leq y \leq x$.) When $0 \leq x \leq 1$, these two constraints are equivalent to $0 \leq y \leq x$ hence

$$\begin{aligned} h(x) &= \int_0^x (y/2)3(x-y)^2 dy \\ &= \int_0^x \left(\frac{3}{2}y^3 - 3xy^2 + \frac{3}{2}x^2 y\right) dy \\ &= \left[\frac{3}{8}y^4 - xy^3 + \frac{3}{4}x^2y^2\right]_0^x \\ &= (1/8)x^4. \end{aligned}$$

When $1 \leq x \leq 2$, the two constraints are equivalent to $x - 1 \leq y \leq x$ and we have

$$\begin{aligned} h(x) &= \int_{x-1}^x (y/2)3(x-y)^2 dy \\ &= \left[\frac{3}{8}y^4 - xy^3 + \frac{3}{4}x^2y^2\right]_{x-1}^x \\ &= \frac{1}{8}x^4 - \left(\frac{3}{8}y^4 - xy^3 + \frac{3}{4}x^2y^2\right)\Big|_{y=x-1} \\ &= \frac{1}{8}x^4 - \frac{3}{8}(x-1)^4 + x(x-1)^3 - \frac{3}{4}x^2(x-1)^2 \\ &= x/2 - 3/8. \end{aligned}$$

When $2 \leq x \leq 3$, the constraints are are equivalent to $x - 1 \leq y \leq 2$ and we have

$$\begin{aligned} h(x) &= \int_{x-1}^{2} (y/2)3(x-y)^2 dy \\ &= \left[\frac{3}{8}y^4 - xy^3 + \frac{3}{4}x^2y^2\right]_{x-1}^{2} \\ &= 6 - 8x + 3x^2 - \left(\frac{3}{8}y^4 - xy^3 + \frac{3}{4}x^2y^2\right)\Big|_{y=x-1} \\ &= 6 - 8x + 3x^2 - (x^4/8 - x/2 + 3/8) \\ &= -x^4/8 + 3x^2 - 15x/2 + 45/8. \end{aligned}$$

When $x < 0$ or $x > 3$ the constraints cannot be satisfied so $h(x) = 0$. Now we compute

$$\begin{aligned} E(X+Y) &= \int_{\mathbb{R}} xh(x)dx \\ &= \int_0^1 x^5/8dx + \int_1^2 x^2/2 - 3x/8dx \\ &\quad + \int_2^3 -x^5/8 + 3x^3 - 15x^2/2 + 45x/8dx \\ &= [x^6/48]_0^1 + [x^3/6 - 3x^2/16]_1^2 \\ &\quad + [-x^6/48 + 3x^4/4 - 5x^3/2 + 45x^2/16]_2^3 \\ &= \frac{1}{48} + \frac{4}{3} - \frac{3}{4} - \frac{1}{6} + \frac{3}{16} \\ &\quad - \frac{729}{48} + \frac{243}{4} - \frac{135}{2} + \frac{405}{16} \\ &\quad + \frac{64}{48} - \frac{48}{4} + \frac{40}{2} - \frac{180}{16} \\ &= 100/48 \\ &= 25/12 \\ &= E(X) + E(Y). \end{aligned}$$

(9) **(3 Points)** Calculate the moment generating functions $g_X(t), g_Y(t)$ for the random variables X, Y from (8). Use your calculation of the density function for $X + Y$ to calculate its moment generating function. Also calculate $g_{X+Y}(t)$ using your knowledge of how this is related to the moment generating functions of X and Y and be sure that you get the same answer.

Solution. Integrate by parts:

$$
\begin{aligned}
g_X(t) &= E(e^{tX}) \\
&= \int_0^2 \frac{xe^{tx}}{2} dx \\
&= \int_0^2 xd\left(\frac{e^{tx}}{2t}\right) \\
&= \left[\frac{x}{2t}e^{tx}\right]_0^2 - \int_0^2 \frac{1}{2t}e^{tx} dx \\
&= \left[\frac{x}{2t}e^{tx}\right]_0^2 - \left[\frac{1}{2t^2}e^{tx}\right]_0^2 \\
&= \frac{1}{t}e^{2t} - \frac{1}{2t^2}e^{2t} + \frac{1}{2t^2}.
\end{aligned}
$$

Similarly, you should find:

$$
\begin{aligned}
g_Y(t) &= \int_0^1 3x^2 e^{tx} dx \\
&= \int_0^1 3x^2 d\left(\frac{e^{tx}}{t}\right) \\
&= \left[\frac{3x^2 e^{tx}}{t}\right]_0^1 - \int_0^1 \frac{6x}{t} e^{tx} dx \\
&= \frac{3e^t}{t} - \int_0^1 6xd\left(\frac{e^{tx}}{t^2}\right) \\
&= \frac{3e^t}{t} - \left[\frac{6xe^{tx}}{t^2}\right]_0^1 - \int_0^1 \frac{6}{t^2}e^{tx} dx \\
&= \frac{3e^t}{t} - \frac{6e^t}{t^2} + \frac{6}{t^2}\left[\frac{e^{tx}}{t}\right]_0^1 \\
&= \frac{3e^t}{t} - \frac{6e^t}{t^2} + \frac{6e^t}{t^3} - \frac{6}{t^3}
\end{aligned}
$$

and

$$
\begin{aligned}
g_{X+Y}(t) &= g_X(t) g_Y(t) \\
&= \left(\frac{1}{t}e^{2t} - \frac{1}{2t^2}e^{2t} + \frac{1}{2t^2}\right)\left(\frac{3e^t}{t} - \frac{6e^t}{t^2} + \frac{6e^t}{t^3} - \frac{6}{t^3}\right).
\end{aligned}
$$

(10) **(2 Points)** How is the density function $f_X(x)$ for a random variable X related to the joint density function $f(x, y)$ for random variables X, Y (not necessarily independent)?

Solution. $f_X(x) = \int_{\mathbb{R}} f(x,y)dy$. Indeed,

$$\begin{aligned} P(a \le X \le b) &= P((X,Y) \in [a,b] \times \mathbb{R}) \\ &= \int_a^b \int_{\mathbb{R}} f(x,y)dxdy \\ &= \int_a^b f_X(x)dx. \end{aligned}$$

(11) **(5 Points)** Calculate the expected number of coin flips needed to observe three heads in a row. Do this by using the theory of absorbing Markov chains. Be sure you clearly indicate (i.e. label) the state space, transition probability matrix, and the fundamental matrix $(\mathrm{Id} - Q)^{-1}$ of your Markov chain.

Solution. Consider the Markov chain with state space $\{T, H, HH, HHH\}$ and transition matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The state HHH is the unique absorbing state. We have

$$\mathrm{Id} - Q = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ -1/2 & 0 & 1 \end{pmatrix}$$

which has inverse

$$(\mathrm{Id} - Q)^{-1} = \begin{pmatrix} 8 & 4 & 2 \\ 6 & 4 & 2 \\ 4 & 2 & 2 \end{pmatrix}$$

so the expected number of flips F is given by

$$\begin{aligned} E(F) &= 1 + (1/2)E(F|\text{initial state is } H) + (1/2)E(F|\text{initial state is } T) \\ &= 1 + (1/2)(8+4+2) + (1/2)(6+4+2) \\ &= 14. \end{aligned}$$

(12) **(5 Points)** A rod of unit length is broken at two randomly chosen points (i.e. by independently and uniformly choosing two numbers in $[0,1]$). What is the probability that the three pieces form a triangle? Now suppose that the rod is broken at one randomly chosen point, then the longer of the two pieces is broken at a randomly chosen point. Now what is the probability that the three pieces form a triangle? *Hint:* Three segments form a triangle iff the length of the longest segment is less than the sum of the lengths of the two shorter segments.

Solution. This is the way I did it: Let $X \in [0,1]$ denote the left break point and Y the right break point, so the lengths of the "left," "middle," and "right" broken pieces are $X, Y - X$, and $1 - Y$ respectively, and (X, Y) is uniformly distributed on the triangle

$$\Delta \ := \ \{(x, y) \in [0, 1]^2 : x \leq y\}.$$

Let $T \subseteq \Delta$ be the set of pairs where the broken pieces form a triangle.

Let $E_L \subseteq \Delta$ be the set of pairs where the left broken piece has the longest length. Then $(x, y) \in E_L$ iff $x \geq y - x$ and $x \geq 1 - y$. Rearranging these inequalities, we see that, for $1/3 \leq x \leq 1/2$ the region E_L is bounded below by the line $y = 1 - x$ and above by the line $y = 2x$ and, for $1/2 \leq x \leq 1$ the region E_L is all of Δ. On E_L, the condition that the broken pieces form a triangle is: $x < 1 - y + y - x$, which is equivalent to $x < 1/2$, so $E_L \cap T$ is the part of E_L where $1/3 \leq x < 1/2$. The area of this region is

$$\begin{aligned} P(E_L \cap T) &= \int_{1/3}^{1/2} 2x - (1 - x) \\ &= \frac{1}{24}, \end{aligned}$$

whereas the rest of E_L (i.e. $E_L \setminus T$) forms a triangle of area 1/8 (draw a picture!). Therefore, we have:

$$\begin{aligned} P(T|E_L) &= \frac{P(T \cap E_L)}{P(E_L)} \\ &= \frac{1/24}{1/24 + 1/8} \\ &= 1/4. \end{aligned}$$

Let $E_M \subseteq \Delta$ be the set of pairs of break points where the middle broken segment has the longest length. Then $(x, y) \in E_M$ iff $y - x \geq x$ and $y - x \geq 1 - y$ iff $y \geq 2x$ and $y \geq x/2 + 1/2$. On $0 \leq x \leq 1/3$, the region E_M is bounded below by the line $y = x/2 + 1/2$ and above by $y = 1$ and on $1/3 \leq x \leq 1/2$, the region E_M is bounded below by the line $y = 2x$ and above by $y = 1$. The region E_M is empty on $x > 1/2$. We find:

$$\begin{aligned} P(E_M) &= \int_0^{1/3} 1 - (x/2 + 1/2)dx + \int_{1/3}^{1/2} 1 - 2xdx \\ &= 1/6. \end{aligned}$$

The region $E_M \cap T$ is the set of $(x, y) \in E_M$ where $y - x < x + 1 - y$ (equivalently: $y < x + 1/2$). Evidently then, $E_M \cap T$ is bounded below by the same lines that bound E_M below, but it is bounded above by the line $y = x + 1/2$ (it helps to draw a picture). We find:

$$\begin{aligned} P(E_M \cap T) &= \int_0^{1/3} (x + 1/2) - (x/2 + 1/2)dx + \int_{1/3}^{1/2} (x + 1/2) - 2xdx \\ &= [x^2/4]_0^{1/3} + [-x^2/x + x/2]_{1/3}^{1/2} \\ &= 1/36 + 1/8 - 1/9. \end{aligned}$$

Now we find:

$$\begin{aligned} P(T|E_M) &= \frac{P(T\cap E_M)}{P(E_M)} \\ &= \frac{1/36+1/8-1/9}{1/6} \\ &= 1/6+3/4-2/3 \\ &= 1/4. \end{aligned}$$

Now, on symmetry grounds (flip the rod around to exchange the left and right broken pieces) we must also have $P(T|E_R) = 1/4$, where $E_R \subset \Delta$ is the region where the right broken piece is the longest. Since $\Delta = E_L \coprod E_M \coprod E_R$ we conclude $P(T) = 1/4$.

The following method is probably more straightforward: Another interpretation of the hint is that the three broken pieces will form a triangle iff the longest segment has length $< 1/2$. Let $X, Y \in [0,1]$ be the position of the first and second cut point. If $X \in [0,1/2]$, then the three broken pieces will form a triangle iff $Y \in [1/2, X+1/2]$ (if $Y < 1/2$ then the rightmost broken segment is longer than $1/2$, while if $Y > X+1/2$ the middle broken segment is longer than $1/2$) . Similarly, if $X \in [1/2,1]$, then the three broken pieces form a triangle iff $Y \in [X-1/2, 1/2]$ (if $Y > 1/2$ then the left broken segment is longer than $1/2$ and if $Y < X - 1/2$ the middle broken segment is longer than $1/2$). The region $T \subseteq [0,1]^2$ of cut points (X,Y) yielding a triangle therefore consists of two triangles, each of area $1/8$. Since the total area of $[0,1]^2$ is 1 and the distribution of (X,Y) is uniform, the desired probability is $1/4$.

If the second cut is always made at random on the longer segment resulting from the first cut, then when X (the first cut) is in $[0,1/2]$, then Y is constrained to lie in $[X,1]$ and when X is in $[1/2,1]$, Y is constrained to lie in $[0,X]$. The cut positions $(X,Y) \in [0,1]^2$ not satisfying these constraints form two triangles: one with vertices $(0,0), (1/2,0), (1/2,1/2)$ and the other with vertices $(1/2,1/2), (1/2,1), (1,1)$. Each of these triangles has area $1/8$, so the cuts of the indicated type are uniformly distributed over a region in $[0,1]^2$ with area $3/4$. All the cuts $(X,Y) \in T$ (i.e. the cuts yielding triangles) satisfy these constraints, so the desired probability is $(1/4)/(3/4) = 1/3$.

(13) **(3 Points)** Adam and Eve fill out cards to order a la carte sushi, but spilled soy sauce has obscured most of their orders as well as any indication of whose card is whose. The only thing legible on the first card is an order for sea urchin and a California roll, and the only thing legible on the second card is an order for sea urchin. You know that Eve likes California rolls but is unlikely to order a sea urchin, but you don't know much about Adam's taste in sushi. Based on this, you estimate that Eve would order a California roll with probability .5 and a sea urchin with probability .1, while Adam would order a California roll with probability .3 and a sea urchin with probability .2. Assume that each person decides to order a given piece of sushi independently from their decisions about other pieces. What is the probability that the first order card is Eve's?

Solution. Let E (resp. A) be the event that the first card is Eve's (resp. Adam's). It is reasonable to assume that, in the absense of any other evidence, the first card is equally likely to be Adam's or Eve's: i.e. $P(E) = .5$ and $P(A) = .5$. Let C_i (resp. S_i) be the event that the i^{th} card has an order for California roll (resp. sea urchin). The evidence we have is $T = C_1 \cap S_1 \cap S_2$. By the assumption about how the choices are made, we have

$$\begin{aligned} P(T|E) &= P(C_1 \cap S_1 \cap S_2|E) \\ &= P(C_1|E)P(S_1|E)P(S_2|E) \\ &= (.5)(.1)(.2) \\ &= .01 \end{aligned}$$

and similarly

$$\begin{aligned} P(T|A) &= P(C_1 \cap S_1 \cap S_2|A) \\ &= P(C_1|A)P(S_1|A)P(S_2|A) \\ &= (.3)(.2)(.1) \\ &= .006. \end{aligned}$$

By Bayes' rule, we find:

$$\begin{aligned} P(E|T) &= \frac{P(T|E)P(E)}{P(T|E)P(E) + P(T|A)P(A)} \\ &= \frac{(.01)(.5)}{(.01)(.5) + (.006)(.5)} \\ &= 5/8. \end{aligned}$$

Now, the incorrect thing to do (which many of you did) is to erroneously assume that the fact that the second card has an order for sea urchin is irrelevant. You would then find (correctly) that $P(E|C_1 \cap S_1) = 5/11$, but this does not take into account all of the evidence (this would be right if the entire second card was illegible). Notice that you would conclude that the first card is more likely to be Adam's, while in fact it is more likely to be Eve's.

(14) **(4 Points)** Calculate the probability that two randomly chosen natural numbers x, y are relatively prime.[1] *Hints:* For $n \in \mathbb{N}$, let E_n be the set of pairs of natural numbers (x, y) where n divides both x and y. Let F_n be the complement of E_n. What is $P(E_n)$? Observe that for distinct primes $p_1, \ldots, p_k$, the events $E_{p_1}, \ldots, E_{p_k}$ are independent, hence so are the events $F_{p_1}, \ldots, F_{p_k}$. Argue that the desired probability is $P(\cap_p F_p)$ as p runs over all primes. The last hint is:

$$\begin{aligned} \zeta(2) &= \sum_{n=1}^{\infty} n^{-2} \\ &= \prod_{\text{primes } p} (1 + p^{-2} + p^{-4} + p^{-6} + \cdots) \\ &= \pi^2/6. \end{aligned}$$

[1]More rigorously: Let p_N be the probability that two numbers chosen independently and uniformly from $\{1, \ldots, N\}$ are relatively prime. The problem is to calculate $\lim_{N\to\infty} p_N$.

Solution. We have $P(E_n) = n^{-2}$ because $E_n = n\mathbb{N} \times n\mathbb{N} \subseteq \mathbb{N}^2$. The events $E_{p_1}, \ldots, E_{p_k}$ are independent because $p_1 \cdots p_k | n$ iff $p_i | n$ for every i, so

$$\begin{aligned} P(E_{p_1} \cap \cdots \cap E_{p_k}) &= P(E_{p_1 \cdots p_k}) \\ &= p_1^{-2} \cdots p_k^{-2} \\ &= P(E_{p_1}) \cdots P(E_{p_k}). \end{aligned}$$

The two numbers (x, y) are relatively prime iff no prime divides both iff $(x, y) \in \cap_p F_p$. By independence of the F_p we have

$$\begin{aligned} P(\cap_p F_p) &= \prod_p P(F_p) \\ &= \prod_p (1 - P(E_p)) \\ &= \prod_p (1 - p^{-2}) \\ &= \prod_p \left(\frac{1}{1 - p^{-2}} \right)^{-1} \\ &= \left(\prod_p \frac{1}{1 - p^{-2}} \right)^{-1} \\ &= \left(\prod_p (1 + p^{-2} + p^{-4} + \cdots) \right)^{-1} \\ &= \zeta(2)^{-1} \\ &= \frac{6}{\pi^2}. \end{aligned}$$