Probability: Midterm 1 Solutions

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(1) **(4 points)** Two fair four sided dice are rolled and the minimum of the two results is recorded. Describe a discrete probability space (Ω, P) and a random variable $X : \Omega \to \mathbb{R}$ that might reasonably be associated with this experiment. Calculate the expected value of the minimum (this should equal $E(X)$ if you have made a "reasonable" choice in the first part of the problem).

Solution. $\Omega = \{(i,j) : i,j \in \{1,2,3,4\}\}$ with uniform density 1/16, $X(i,j) := \min(i,j)$. By counting the number of such ordered pairs with minimum 1, 2, 3, 4, we find $P(X=1) = 7/16$, $P(X=2) = 5/16$, $P(X=3) = 3/16$, $P(X=4) = 1/16$, hence

$$\begin{aligned} E(X) &= (1)(7/16) + (2)(5/16) + (3)(3/16) + (4)(1/16) \\ &= 15/8. \end{aligned}$$

(2) **(5 points)** Two fair dice, one with k sides (labelled $1, 2, \ldots, k$), and one with m sides (labelled $1, 2, \ldots, m$), are rolled n times (each). Let X denote the total number of 1's rolled. Calculate the expected value $E(X)$ and variance $V(X)$. What is Chebyshev's upper bound for the probability that X will differ from $E(X)$ by at least 2?

Solution. These are just two independent experiments. Let X_1 denote the number of 1's from the k sided die and X_2 denote the number of 1's from the m sided die. X_1 and X_2 are binomially distributed with n trials and success probability $1/k$ and $1/m$ respectively. Since $X = X_1 + X_2$ and the X_i are independent,

$$\begin{aligned} E(X) &= E(X_1) + E(X_2) \\ &= n/k + n/m \\ V(X) &= V(X_1) + V(X_2) \\ &= n\left(\frac{k-1}{k^2} + \frac{m-1}{m^2}\right). \end{aligned}$$

Chebyshev says

$$\begin{aligned} P(|X - E(X)| \geq 2) &\leq \frac{V(X)}{2^2} \\ &= \frac{n}{4}\left(\frac{k-1}{k^2} + \frac{m-1}{m^2}\right). \end{aligned}$$

(3) **(4 points)** The populations of **A**tlanta, **S**eattle, and **R**eno are 500 thousand, 600 thousand, and 400 thousand, respectively. Suppose we know only that Walter is from one of these three cities. What is the probability that Walter is from Atlanta? Nine out of ten Atlantans prefer **C**oke to **P**epsi and two out of three

Seattlites prefer Pepsi to Coke, while Reno residents are evenly split in their preferences.[1] Suppose we also learn that Walter prefers Coke to Pepsi. Now what is the probability that Walter is from Atlanta?

Solution. The initial probability that Walter is from Atlanta is 500 thousand in 1.5 million, or 1/3. If he prefers Coke, then by Bayes' rule,

$$\begin{aligned} P(A|C) &= \frac{P(A)P(C|A)}{P(A)P(C|A) + P(S)P(C|S) + P(R)P(C|R)} \\ &= \frac{(1/3)(9/10)}{(1/3)(9/10) + (6/15)(1/3) + (4/15)(1/2)} \\ &= \frac{3/10}{3/10 + 2/15 + 2/15} \\ &= \frac{45/150}{85/150} \\ &= \frac{45}{85} \\ &= \frac{9}{17}. \end{aligned}$$

(4) **(4 points)** What is the probability of being dealt a flush in poker? That is: 5 cards are chosen at random from a standard 52 card deck with 4 suits ($\spadesuit$, $\heartsuit$, $\diamondsuit$, $\clubsuit$) and 13 cards (with "numerical values" 2,3,...,10,J,Q,K,A) per suit. What is the probability that all 5 cards are of the same suit? What is the probability of being dealt a full house (three of the five cards have the same numerical value and the other two cards also have the same numberical value, e.g. $3\heartsuit, 3\diamondsuit, 3\spadesuit, A\clubsuit, A\spadesuit$ is a full house)?

Solution. There are four possible suits, and $\binom{13}{5}$ flushes in any specified suit, so the number of hands that are flushes is

$$4\binom{13}{5}.$$

Since there are $\binom{52}{5}$ possible hands, each equally likely, the probability of being dealt a flush is

$$\frac{4\binom{13}{5}}{\binom{52}{5}}.$$

When it comes to full houses, there are 13 possible numerical values to have 3 of and 13 possible numerical values to have 2 of (these obviously must be distinct, but they are "distinguishable"), and there are 4 ways to choose 3 of the 4 cards

[1] Assume everyone has a preference.

with the same numerical value and 6 ways to choose 2 of the 4 cards with the same numerical value, so the total number of full houses is

$$13 \cdot 12 \cdot 4 \cdot 6,$$

and the probability of being dealt a full house is

$$\frac{13 \cdot 12 \cdot 4 \cdot 6}{\binom{52}{5}}.$$

(5) **(3 points)** The $\binom{n}{2}$ edges $\{\{i,j\} : 1 \leq i < j \leq n\}$ of the complete graph K_n with vertices $[n] = \{1, \ldots, n\}$ are colored red and blue (with equal probability) at random. Recall that $A \subseteq [n]$ is called *monochromatic* if each of the $\binom{|A|}{2}$ edges between vertices of A is colored the same color. For $A, B \subseteq [n]$, give a criterion in terms of $|A \cap B|$ for the events "A is monochromatic" and "B is monochromatic" to be independent.

Solution. These events are obviously independent if $|A \cap B| \leq 1$, for then the set of edges between vertices of A is disjoint from the set of edges between vertices of B. Suppose $|A \cap B| = 2$, so $A \cap B = \{i,j\}$ for some distinct $i, j \in [n]$. The edge $\{i,j\}$ is the unique edge whose vertices are in both A and B. Then the probability that A is monochromatic given B is monochromatic (all blue, say) is just the probability that the other $\binom{|A|}{2} - 1$ edges of A have the same color as the edge $\{i,j\}$ (blue in this case), which is $(1/2)$ to the power $\binom{|A|}{2} - 1$. Rather coincidentally, this is the same thing as the probability that A is monochromatic, so in this case the events are independent. If $|A \cap B| > 2$, then A and B have multiple edges in common, so if B is monochromatic, then we know that several edges of A are the same color (the ones shared with B) and the probability that A is monochromatic will be larger than usual, hence the events are not independent in this case.

(6) **(4 points)** The number of accidents (per month) at a certain factory has a Poisson distribution. If the probability that there is *at least one* accident is 1/2, what is the probability that there are exactly two accidents?

Solution. If λ is the expected number of accidents in a month, then the probability that there are k accidents in a month is

$$P(k) \quad = \quad e^{\lambda}\lambda^k/k!.$$

Since the probability of having at least one accident is $1/2$, we have

$$\begin{aligned} 1/2 &= P(k \geq 1) \\ &= 1 - P(k = 0) \\ &= 1 - e^{-\lambda}, \end{aligned}$$

which implies $\lambda = \ln 2$, so

$$\begin{aligned} P(2) &= e^{-\ln 2}\frac{(\ln 2)^2}{2!} \\ &= \frac{(\ln 2)^2}{4}. \end{aligned}$$

(7) **(4 points)** A pair of dice is rolled until either a 4 is rolled (the numbers on the two dice add up to 4) or a 7 is rolled. What is the expected number of rolls needed? (This sort of computation arises if one wishes to compute the expected number of dice rolls per craps game.)

Solution. We are just asking for the expected number of trials until the first "success" when the probability p of success is the probability of rolling either a 4 or a 7; this is the expected value of the geometric distribution, which is $1/p$. We have seen many times that $p = 3/36 + 6/36 = 1/4$ in this case, so the answer is 4.

(8) **(3 points)** State the Weak Law of Large Numbers.

Solution. Look in the book or your notes.