Probability: Midterm 2 Solutions

Fall, 2009
Instructor: W. D. Gillam

(1) **(3 points)** Give the definitions of the beta and gamma functions, the relationship between them, and the functional equation for the gamma function. (No proofs required.)

(2) **(4 points)** Give the density function for a random variable X with normal distribution with expected value μ and variance σ^2. Express $P(a \leq X \leq b)$ as a definite integral of the standard normal density function.

Solution. $(X-\mu)/\sqrt{\sigma^2}$ has a standard normal distribution, so

$$\begin{aligned} P(a \leq X \leq b) &= P(\frac{a-\mu}{\sqrt{\sigma^2}} \leq \frac{X-\mu}{\sqrt{\sigma^2}} \leq \frac{b-\mu}{\sqrt{\sigma^2}}) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sqrt{\sigma^2}}}^{\frac{b-\mu}{\sqrt{\sigma^2}}} e^{-x^2/2} dx. \end{aligned}$$

(3) **(6 points)** Give the density functions and expected values for the gamma, beta, and Student distributions. Indicate the choices of the parameters α, β for which the gamma density function specializes to the chi squared density function with n degrees of freedom. Explain how the chi squared distribution and Student's distribution arise from the normal distribution.

(4) **(4 points)** Suppose X has a binomial distribution based on n trials with success probability p. Calculate the moment generating function $g_X(t)$ of X. *Hint:* You can use multiplicativity of MGFs under sums of independent random variables to reduce to the case $n = 1$.

Solution. When $n = 1$, this is $g_X(t) = E(e^{tX}) = pe^{t\cdot 1} + qe^{t\cdot 0} = pe^t + q$, so in general it is $g_X(t) = (pe^t + q)^n$.

(5) **(4 points)** Suppose X has a beta distribution with parameters $\alpha = \beta = 1/2$. Find the cumulative distribution function $F_X(x)$ of X.

Solution. The density function for X is given by

$$\begin{aligned} f(y) &= \frac{\Gamma(1)}{\Gamma(1/2)^2} y^{-1/2}(1-y)^{-1/2} \\ &= \frac{1}{\pi} \frac{1}{\sqrt{y}\sqrt{1-y}} \end{aligned}$$

(supported on $[0,1]$). For x with $0 \le x \le 1$, we compute

$$\begin{aligned} \frac{1}{\pi}\int_0^x \frac{1}{\sqrt{y}\sqrt{1-y}}dy &= \left[\frac{2}{\pi}\arcsin\sqrt{y}\right]_0^x \\ &= \frac{2}{\pi}\arcsin\sqrt{x}, \end{aligned}$$

so

$$F_X(x) = \begin{cases} 0, & x \le 0 \\ (2/\pi)\arcsin\sqrt{x}, & 0 \le x \le 1 \\ 1, & x \ge 1 \end{cases}$$

(6) **(4 points)** Suppose X and Y are IIDRV, each with density function

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

Calculate the density function of $X+Y$.

Solution. The desired density is given by

$$g(x) = \int_{-\infty}^{\infty} f(y)f(x-y)dy.$$

The integrand will vanish unless $0 \le y \le 1$ and $0 \le x-y \le 1$. The second inequality is equivalent to $x-1 \le y \le x$. So the integrand is zero if $x<0$ or if $x>2$. If $0 \le x \le 1$ it is given by

$$\begin{aligned} g(x) &= \int_0^x (2y)(2(x-y))dy \\ &= \int_0^x 4xy - 4y^2 dy \\ &= \left[2xy^2 - (4/3)y^3\right]_0^x \\ &= (2/3)x^3. \end{aligned}$$

When $1 \le x \le 2$, it is given by

$$\begin{aligned} g(x) &= \int_{x-1}^1 4xy - 4y^2 \\ &= \left[2xy^2 - (4/3)y^3\right]_{x-1}^1 \\ &= 2x - 4/3 - 2x(x-1)^2 + (4/3)(x-1)^3. \end{aligned}$$

(7) **(4 points)** A point P is chosen randomly (uniformly) along a thin rod (a coat hanger, say). The rod is then bent to form a right angle at P, so that the bent rod forms the two shorter sides of a right triangle. Let θ denote the smallest angle in this triangle. Find the expected value of $\tan\theta$.

Solution. By using the appropriate units, we may assume the length of the rod is 1, so that P is uniformly distributed on $[0,1]$. Then the two shorter sides of the right triangle will have lengths P and $1-P$. The tangents of the two angles will then be $P/(1-P)$ and $(1-P)/P$; the one of these that is at most 1 is the tangent of the smaller angle, so if $P \geq 1/2$, $\tan\theta = (1-P)/P$, while if $P \leq 1/2$, $\tan\theta = P/(1-P)$. We have:

$$\begin{aligned}
E(\tan\theta) &= \int_0^{1/2} \frac{P}{1-P} dP + \int_{1/2}^1 \frac{1-P}{P} dP \\
&= \int_0^{1/2} -1 + \frac{1}{1-P} dP + \int_{1/2}^1 \frac{1}{P} - 1 dP \\
&= -1/2 + [-\ln(1-P)]_0^{1/2} - 1/2 + [\ln P]_{1/2}^1 \\
&= -1/2 - \ln(1/2) - 1/2 - \ln(1/2) \\
&= 2\ln 2 - 1.
\end{aligned}$$

(8) **(4 points)** A random variable X is said to have a *Weibull distribution* with parameters $\alpha, m > 0$ if

$$f(x) = \begin{cases} \frac{mx^{m-1}}{\alpha} \exp\left(-\frac{x^m}{\alpha}\right), & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

is a density for X. Calculate the expected value and variance of such an X. (Express your answer in terms of gamma functions.)

Solution.

$$E(X) = \int_0^\infty \frac{mx^m}{\alpha} \exp\left(-\frac{x^m}{\alpha}\right).$$

Make the substitution $y = x^m/\alpha$, so $x = \alpha^{1/m} y^{1/m}$. This becomes:

$$\begin{aligned}
E(X) &= \int_0^\infty \alpha^{1/m} y^{1/m} e^{-y} dy \\
&= \alpha^{1/m} \Gamma(\frac{m+1}{m}).
\end{aligned}$$

Similarly,

$$\begin{aligned}
V(X) &= \int_0^\infty \frac{mx^{m+1}}{\alpha} \exp\left(-\frac{x^m}{\alpha}\right) dx - E(X)^2 \\
&= \int_0^\infty \alpha^{2/m} y^{2/m} e^{-y} dy - \alpha^{2/m} \Gamma(\frac{m+1}{m})^2 \\
&= \alpha^{2/m} (\Gamma(\frac{m+2}{m}) - \Gamma(\frac{m+1}{m})^2).
\end{aligned}$$