Probability: Problem Set 9

Fall 2009
Instructor: W. D. Gillam
Due Dec. 4, start of class

Instructions. Print your name in the upper right corner of the paper and write "Problem Set 9" on the first line on the left. Skip a few lines. When you finish this, indicate on the second line on the left the amount of time you spent on this assignment and rate its difficulty on a scale of $1 - 5$ ($1 =$ easy, $5 =$ hard). When you are asked to plot graphs, please use a computer and reasonable plotting software (Mathematica, Matlab, etc.)

This problem set is concerned with *Markov chains*, named for the Soviet mathematician Andrei Andreevich Markov (1856-1922).

(1) Recall the negative binomial distribution which counts the number of trials of a two outcome ($0, 1$ say, with probabilities $1 - p, p$) experiment up to and including the n^{th} success. Explain how to view this as a (finite, homogeneous, absorbing) Markov chain with state space $X = \{0, 1, \dots, n\}$. What probability distribution μ should you use for the "initial" probabilities? What is... ...the transition matrix P? ...the matrix Q of transition probabilities between transient states? ...the fundamental matrix $(\mathrm{Id} - Q)^{-1}$? Use the general theory to compute the expected number of trials needed to reach the n^{th} success.

(2) Recall that we discussed the problem of finding the expected number of fouls needed for a player to make some number n of free throws, assuming that the player shoots two free throws each time (s)he (in the (W)NBA) is fouled. This depends on the player's free throw percentage p, which for Shaq is $p = .528$. Explain how this problem can be viewed as a (finite, homogeneous, absorbing) Markov chain with state space $X = \{0, 1, \dots, n\}$. Unlike in the previous problem, there will be a positive probability of moving from state i to $i+2 \leq n$, which makes the transition matrix P less sparse, and therefore harder to invert. Use the general machinery to compute the expected number of fouls needed to make $n = 1, 2, 3$ free throws (as a function of p). Plot these three functions of p on the same axes. This might be hard if you are not capable of using some computer software to invert the matrices... actually you only have to invert an upper triangular 3×3 matrix, which you should be able to do by hand...

(3) In class we studied the expected number of coin flips needed to observe two heads in a row: "HH". One can of course also consider the expected number of coin flips needed to observe some more elaborate pattern: "HHTH," say. This can be calculated in a variety of ways. Do Exercises 28 and 30 in Chapter 11.2 of Grinstead and Snell (pp. 428-430) to get the general idea.

(4) In class I left you two little exercises concerning the harmonic numbers

$$H_n \quad := \quad 1 + 1/2 + 1/3 + \cdots + 1/n.$$

The first is to show that

$$H_n \quad = \quad 1 + (1/n)(H_{n-1} + H_{n-2} + \cdots + H_1 + H_0)$$

(for $n > 0$). The second is to show that the function

$$b(x) \quad := \quad -\frac{\ln(1-x)}{1-x}$$

satisfies

$$(1-x)^2 b'(x) - (1-x)b(x) - 1 = 0.$$

(and has $b(0) = 0$; it is then the unique function with these two properties by the fundamental theorem of ODEs). Recall that we showed in class (in at least two different ways) that

$$b(x) \quad = \quad \sum_{n=1}^{\infty} H_n x^n$$

(as formal series).