

Probability: Problem Set 9 Solutions

Fall 2009

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- (1) Recall the negative binomial distribution which counts the number of trials of a two outcome (0, 1 say, with probabilities $1 - p, p$) experiment up to and including the n^{th} success. Explain how to view this as a (finite, homogeneous, absorbing) Markov chain with state space $X = \{0, 1, \dots, n\}$. What probability distribution μ should you use for the “initial” probabilities? What is... ..the transition matrix P ? ...the matrix Q of transition probabilities between transient states? ...the fundamental matrix $(\text{Id} - Q)^{-1}$? Use the general theory to compute the expected number of trials needed to reach the n^{th} success.

Solution. I think of the states as being the number of successes up to a given time point, so we should set $\mu(i)$ equal to zero unless $i = 0$ (the process starts from zero successes). The transition matrix P is then

$$P = \begin{pmatrix} q & p & 0 & 0 & \cdots & 0 & 0 \\ 0 & q & p & 0 & \cdots & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

The matrix $(\text{Id} - Q)$ is the $n \times n$ (really: $\{0, \dots, n - 1\} \times \{0, \dots, n - 1\}$) matrix

$$\text{Id} - Q = \begin{pmatrix} p & -p & 0 & 0 & \cdots & 0 & 0 \\ 0 & p & -p & 0 & \cdots & 0 & 0 \\ 0 & 0 & p & -p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p & -p \\ 0 & 0 & 0 & 0 & \cdots & 0 & p \end{pmatrix}$$

which has inverse

$$(\text{Id} - Q)^{-1} = \begin{pmatrix} p^{-1} & p^{-1} & p^{-1} & p^{-1} & \cdots & p^{-1} & p^{-1} \\ 0 & p^{-1} & p^{-1} & p^{-1} & \cdots & p^{-1} & p^{-1} \\ 0 & 0 & p^{-1} & p^{-1} & \cdots & p^{-1} & p^{-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p^{-1} & p^{-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & p^{-1} \end{pmatrix}$$

The expected number of trials to reach the unique absorbing state n from the transient state 0 is then given by the sum of the entries in the first row (really the 0^{th} row), which is np^{-1} .

- (2) Recall that we discussed the problem of finding the expected number of fouls needed for a player to make some number n of free throws, assuming that the player shoots two free throws each time (s)he (in the (W)NBA) is fouled. This depends on the player’s free throw percentage p , which for Shaq is $p = .528$.

Explain how this problem can be viewed as a (finite, homogeneous, absorbing) Markov chain with state space $X = \{0, 1, \dots, n\}$. Unlike in the previous problem, there will be a positive probability of moving from state i to $i + 2 \leq n$, which makes the transition matrix P less sparse, and therefore harder to invert. Use the general machinery to compute the expected number of fouls needed to make $n = 1, 2, 3$ free throws (as a function of p). Plot these three functions of p on the same axes. This might be hard if you are not capable of using some computer software to invert the matrices.

Solution. The player's probabilities of making 0, 1, 2 free throws on one foul are given by

$$\begin{aligned} p_0 &:= (1-p)^2 \\ p_1 &:= 2p(1-p) \\ p_2 &:= p^2, \end{aligned}$$

respectively. Note

$$p_0 + p_1 + p_2 = 1.$$

Here I view the states as the number of free throws made up to a given point in time, so that the transition matrix is

$$P = \begin{pmatrix} p_0 & p_1 & p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_0 & p_1 & p_2 & \cdots & 0 & 0 \\ 0 & 0 & p_0 & p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_1 & p_2 \\ 0 & 0 & 0 & 0 & \cdots & p_0 & p_1 + p_2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

and the matrix $(\text{Id} - Q)$ is

$$\begin{aligned} (\text{Id} - Q) &= \begin{pmatrix} p_1 + p_2 & -p_1 & -p_2 & 0 & \cdots & 0 & 0 \\ 0 & p_1 + p_2 & -p_1 & -p_2 & \cdots & 0 & 0 \\ 0 & 0 & p_1 + p_2 & -p_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p_1 + p_2 & -p_1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & p_1 + p_2 \end{pmatrix} \\ &= \begin{pmatrix} 2p - p^2 & -2p + 2p^2 & -p^2 & 0 & \cdots & 0 & 0 \\ 0 & 2p - p^2 & -2p + 2p^2 & -p^2 & \cdots & 0 & 0 \\ 0 & 0 & 2p - p^2 & -2p + 2p^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2p - p^2 & -2p + 2p^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 2p - p^2 \end{pmatrix} \end{aligned}$$

The case $n = 1$ is rather trivial: the matrix $\text{Id} - Q$ is the 1×1 matrix with entry $p_1 + p_2$, so its inverse is just $(p_1 + p_2)^{-1}$. The expected number of fouls in this case

is

$$\begin{aligned} E_1 &= (p_1 + p_2)^{-1} \\ &= \frac{1}{2p - p^2}, \end{aligned}$$

which we already know since this may be viewed as the expectation of a random variable with geometric distribution with success probability $p_1 + p_2$. The fundamental matrix for the case $n = 2$ is given by

$$\begin{aligned} (\text{Id}_2 - Q_2)^{-1} &= \begin{pmatrix} p_1 + p_2 & -p_1 \\ 0 & p_1 + p_2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{p_1 + p_2} & \frac{p_1}{(p_1 + p_2)^2} \\ 0 & \frac{1}{p_1 + p_2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2p - p^2} & \frac{2p(1 - p)}{(2p - p^2)^2} \\ 0 & \frac{1}{(2p - p^2)^2} \end{pmatrix} \end{aligned}$$

According to the general theory, the expected number of trials (fouls) needed to reach the unique absorbing state 2 (i.e. to make 2 free throws) is given by summing the entries in the first row:

$$E_2 = \frac{4 - 3p}{4p - 4p^2 + p^3}.$$

For $n = 3$, the fundamental matrix is given by

$$\begin{aligned} (\text{Id}_3 - Q_3)^{-1} &= \begin{pmatrix} p_1 + p_2 & -p_1 & -p_2 \\ 0 & p_1 + p_2 & -p_1 \\ 0 & 0 & p_1 + p_2 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \frac{1}{p_1 + p_2} & \frac{p_1^2 + p_1 p_2}{(p_1 + p_2)^3} & \frac{p_1^2 + p_1 p_2 + p_2^2}{(p_1 + p_2)^3} \\ 0 & \frac{1}{p_1 + p_2} & \frac{p_1^2 + p_1 p_2}{(p_1 + p_2)^3} \\ 0 & 0 & \frac{1}{p_1 + p_2} \end{pmatrix} \\ &= \frac{1}{D(p)} \begin{pmatrix} 4 - 4p + p^2 & 4 - 6p + 2p^2 & 4 - 6p + 3p^2 \\ 0 & 4 - 4p + p^2 & 4 - 6p + 2p^2 \\ 0 & 0 & 4 - 4p + p^2 \end{pmatrix}, \end{aligned}$$

where

$$D(p) = 8p - 12p^2 + 6p^3 - p^4.$$

As before, the expected number of fouls needed to make 3 free throws is given by summing the entries in the first row:

$$E_3 = \frac{12 - 16p + 6p^2}{8p - 12p^2 + 6p^3 - p^4}.$$

For $n = 4$, the fundamental matrix is given by

$$\begin{aligned}
 (\text{Id}_4 - Q_4)^{-1} &= \begin{pmatrix} p_1 + p_2 & -p_1 & -p_2 & 0 \\ 0 & p_1 + p_2 & -p_1 & -p_2 \\ 0 & 0 & p_1 + p_2 & -p_1 \\ 0 & 0 & 0 & p_1 + p_2 \end{pmatrix}^{-1} \\
 &= \begin{pmatrix} \frac{1}{p_1 + p_2} & \frac{p_1^3 + 2p_2p_1^2 + p_2^2p_1}{(p_1 + p_2)^4} & \frac{p_1^3 + 2p_2p_1^2 + 2p_2^2p_1 + p_2^3}{(p_1 + p_2)^4} & \frac{p_1^3 + 2p_2p_1^2 + 2p_2^2p_1}{(p_1 + p_2)^4} \\ 0 & \frac{1}{p_1 + p_2} & \frac{p_1^3 + 2p_2p_1^2 + p_2^2p_1}{(p_1 + p_2)^4} & \frac{p_1^3 + 2p_2p_1^2 + 2p_2^2p_1 + p_2^3}{(p_1 + p_2)^4} \\ 0 & 0 & \frac{1}{p_1 + p_2} & \frac{p_1}{(p_1 + p_2)^2} \\ 0 & 0 & 0 & \frac{1}{p_1 + p_2} \end{pmatrix} \\
 &= \frac{1}{D_4(p)} \begin{pmatrix} 8 - 12p + 6p^2 - p^3 & 8 - 16p + 10p^2 - 2p^3 & 8 - 16p + 12p^2 - 3p^3 & 8 - 16p + 12p^2 - 4p^3 \\ 0 & 8 - 12p + 6p^2 - p^3 & 8 - 16p + 10p^2 - 2p^3 & 8 - 16p + 12p^2 - 3p^3 \\ 0 & 0 & 8 - 12p + 6p^2 - p^3 & 8 - 16p + 10p^2 - 2p^3 \\ 0 & 0 & 0 & 8 - 12p + 6p^2 - p^3 \end{pmatrix},
 \end{aligned}$$

where

$$D_4(p) = 16p - 32p^2 + 24p^3 - 8p^4 + p^5.$$

Adding the entries in the first row, we find:

$$E_4 = \frac{32 - 60p + 40p^2 - 10p^3}{16p - 32p^2 + 24p^3 - 8p^4 + p^5}.$$

- (3) In class we studied the expected number of coin flips needed to observe two heads in a row: “HH”. One can of course also consider the expected number of coin flips needed to observe some more elaborate pattern: “HHTH,” say. This can be calculated in a variety of ways. Do Exercises 28 and 30 in Chapter 11.2 of Grinstead and Snell (pp. 428-430) to get the general idea.

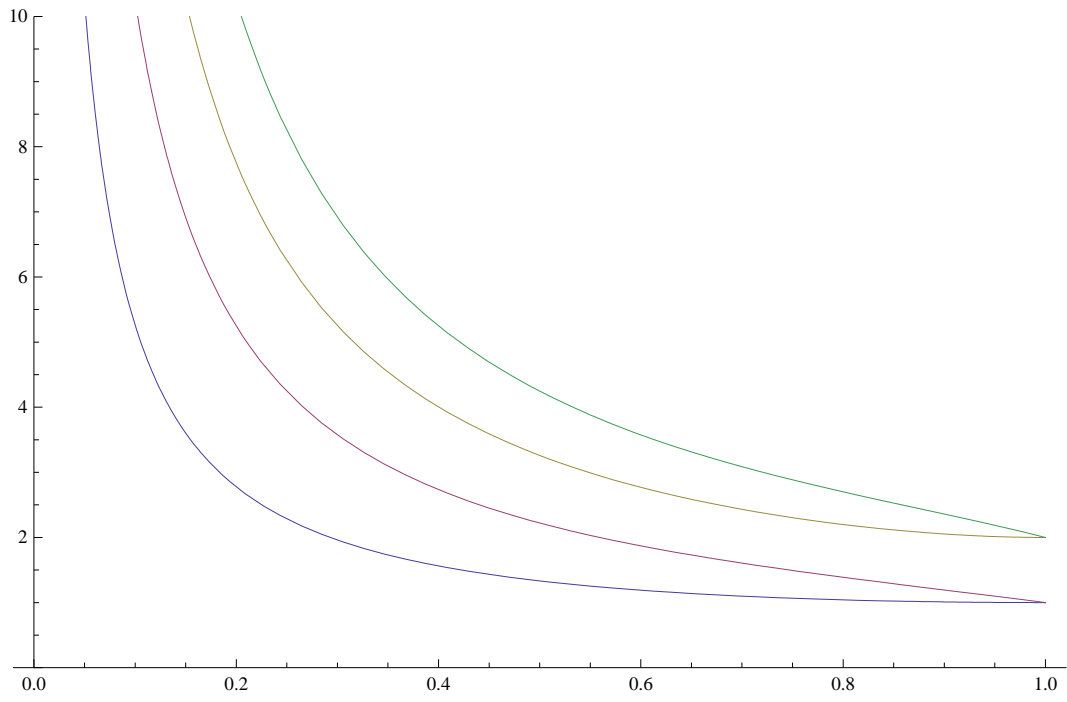


FIGURE 1. Graphs of E_1, E_2, E_3, E_4 in blue, purple, yellow, green, respectively.