

Algebraic Geometry Problems

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Warning: Difficulty varies.

- (1) Let $f : X \rightarrow Y$ be a morphism of schemes, $\iota : Z \hookrightarrow Y$ a closed subscheme of Y with ideal sheaf \mathcal{I} . Let

$$f^{-1}\mathcal{I} := f^{-1}\mathcal{I} \cdot \mathcal{O}_X \subseteq \mathcal{O}_X$$

be the inverse image ideal sheaf (Hartshorne, page 163).

- (a) $f^{-1}\mathcal{I}$ is the ideal sheaf of the closed subscheme obtained by base-change:

$$Z' := Z \times_Y X \hookrightarrow X.$$

- (b) For any $n \in \mathbb{Z}_{\geq 0}$, the ideal sheaves $f^{-1}(\mathcal{I}^n)$ and $(f^{-1}\mathcal{I})^n$ are the same. The closed subscheme Z_{n+1} of Y defined by the ideal sheaf \mathcal{I}^{n+1} is called the n^{th} infinitesimal neighborhood of Z in Y .
- (c) Blowing up Z' and using the universal property of the fibered product and blow-up (see Corollary 7.15 in Hartshorne) we get a commutative diagram with cartesian square:

$$\begin{array}{ccccc} \text{Bl}_{Z'}X & \xrightarrow{g} & \text{Bl}_Z Y \times_Y X & \longrightarrow & \text{Bl}_Z Y \\ & & \downarrow & & \downarrow \\ & & X & \xrightarrow{f} & Y \end{array}$$

Show that g is a closed immersion.

- (d) Show that

$$\text{Bl}_Z Y \times_Y X \cong \text{Proj} \bigoplus_n f^*(\mathcal{I}^n)$$

and that, under this identification, the ideal sheaf of the closed immersion g is (the sheaf corresponding to the graded module) $\bigoplus_n \mathbf{L}^1 f^* \mathcal{O}_{Z_n}$.

- (2) Let

$$\begin{array}{ccc} W & \xrightarrow{j} & X \\ g \downarrow & & \downarrow f \\ Z & \xrightarrow{i} & Y \end{array}$$

be a cartesian diagram of schemes where i (hence j) is a closed immersion and f (hence g) is flat. Let \mathcal{I}, \mathcal{J} be the ideal sheaves of i, j respectively. Observe that $g^*\mathcal{I}^n = \mathcal{J}^n$ for all n . In particular, the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ of Z in Y pulls back (via g) to the conormal sheaf $\mathcal{J}/\mathcal{J}^2$ of W in X . The diagram of normal cones

$$\begin{array}{ccc} C_{W/X} & \longrightarrow & W \\ \downarrow & & \downarrow \\ C_{Z/Y} & \longrightarrow & Z \end{array}$$

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is cartesian. *Hint:* See Exercise (1a).

(3) **Gluing along closed subschemes I.** Let A be a ring.

(a) Show that the category of A -algebras has *cartesian* squares

$$\begin{array}{ccc} B_1 \times_C B_2 & \xrightarrow{\pi_2} & B_2 \\ \pi_1 \downarrow & & \downarrow f_2 \\ B_1 & \xrightarrow{f_1} & C \end{array}$$

where the underlying set of the cartesian product $B_1 \times_C B_2$ agrees with the cartesian product of the underlying sets:

$$B_1 \times_C B_2 = \{(b_1, b_2) : f_1(b_1) = f_2(b_2)\}$$

Let $h = \pi_1 f_1 = \pi_2 f_2 : B_1 \times_C B_2 \rightarrow C$ denote the natural map. If you think about it, this is really just an incarnation of the fact that inverse limits commute amongst themselves and rings are defined in terms of inverse limits.

An analogous statement holds for ring *objects* in any category.

(b) Prove that the formation of $B_1 \times_C B_2$ commutes with filtered direct limits, in the sense that if

$$\begin{array}{ccc} & & (B_2)_i \\ & & \downarrow \\ (B_1)_i & \longrightarrow & (C)_i \end{array}$$

is a diagram of filtered direct limit systems of rings, then the natural map

$$\varinjlim (B_1 \times_C B_2)_i \rightarrow \varinjlim (B_1)_i \times \varinjlim (C)_i \varinjlim (B_2)_i$$

is an isomorphism. Note that this is just a property of the category of sets; in fact it holds in any topos.

(c) In particular, conclude that formation of $B_1 \times_C B_2$ commutes with localization at a multiplicative subset S of C , in the sense that the map

$$\begin{aligned} (h^{-1}S)^{-1}(B_1 \times_C B_2) &\rightarrow (f_1^{-1}S)^{-1}B_1 \times_{S^{-1}C} (f_2^{-1}S)^{-1}B_2 \\ \frac{(b_1, b_2)}{(s_1, s_2)} &\mapsto \left(\frac{b_1}{s_1}, \frac{b_2}{s_2} \right) \end{aligned}$$

is a (well-defined) isomorphism. (Hopefully you see how this is related to the previous part.)

(d) Let $(b_1, b_2) \in (B_1 \times_C B_2)$ and let $c := f_1(b_1) = f_2(b_2)$ be its image in C . Show that

$$\begin{array}{ccc} (B_1 \times_C B_2)_{(b_1, b_2)} & \xrightarrow{\pi_2} & (B_2)_{b_2} \\ \pi_1 \downarrow & & \downarrow f_2 \\ (B_1)_{b_1} & \xrightarrow{f_1} & C_c \end{array}$$

is cartesian.

(e) Show that, if f_1 and f_2 are surjective, then so are π_1 and π_2 , and the diagram

$$\begin{array}{ccc} B_1 \times_C B_2 & \xrightarrow{\pi_2} & B_2 \\ \pi_1 \downarrow & & \downarrow f_2 \\ B_1 & \xrightarrow{f_1} & C \end{array}$$

is also cocartesian. That is, the natural map $B_1 \otimes_{B_1 \times_C B_2} B_2 \rightarrow C$ is an isomorphism. *Hint:* Show that C has the universal property of the tensor product; by surjectivity there is only one possible way to find the required lift, so you just have to check that it works!

(f) Show that $\text{Ker } \pi_1 = \{0\} \times \text{Ker } f_2$ and $\text{Ker } \pi_2 = \text{Ker } f_1 \times \{0\}$ and

$$\text{Ker}(B_1 \times_C B_2 \rightarrow C) = \text{Ker } f_1 \times \text{Ker } f_2.$$

(g) Assuming f_1, f_2 surjective, show that the natural (continuous!) map

$$\text{Spec } B_1 \coprod_{\text{Spec } C} \text{Spec } B_2 \rightarrow \text{Spec}(B_1 \times_C B_2)$$

is a homeomorphism of topological spaces. The coproduct on the left is the pushout in the category of topological spaces, and for now this is just a statement about topological spaces, but see the next exercise for a statement about ringed spaces. *Hint:* Use the fact that each $\text{Spec } B_i \rightarrow \text{Spec}(B_1 \times_C B_2)$ is closed to show this map is closed, so to show it is a homeomorphism it is enough to show it is bijective. Now, if P is a prime ideal of $B_1 \times_C B_2$, show that it contains at least one of $\{0\} \times \text{Ker } f_2$ and $\text{Ker } f_1 \times \{0\}$, so by (3f) P is in the image of at least one of the maps $\text{Spec } B_i \rightarrow \text{Spec}(B_1 \times_C B_2)$, and in the image of both iff it contains both of these ideals iff it contains their sum iff it is in the image of $\text{Spec } C \rightarrow \text{Spec}(B_1 \times_C B_2)$ (second part of (3f)).

(4) **Gluing along closed subschemes II.** Recall that a morphism $f : Z \rightarrow X$ of ringed spaces is called a *closed embedding* if the underlying map of topological spaces is a closed embedding and the map

$$f_b : \mathcal{O}_X \rightarrow f_* \mathcal{O}_Z$$

is a surjection of sheaves on X (not merely an epimorphism in the category of sheaves of rings on X). The kernel \mathcal{I} of this surjection is an ideal of \mathcal{O}_X called the *ideal* (or *ideal sheaf*) of Z .

(a) Show that a closed embedding is a monomorphism in the category of ringed spaces.

(b) Suppose

$$\begin{array}{ccc} Z & \xrightarrow{i_1} & X_1 \\ \downarrow i_2 & & \\ X_2 & & \end{array}$$

is a diagram of closed embeddings of ringed spaces. Show that it has a pushout $X_1 \coprod_Z X_2$ in the category of ringed spaces. The underlying topological space of the pushout is the pushout of the underlying topological spaces, and its sheaf of rings is the cartesian product

$$\mathcal{O}_{X_1} \times_{\mathcal{O}_Z} \mathcal{O}_{X_2}$$

(omitting notation for pushforward of sheaves). Note that the cartesian product of sheaves of rings agrees with the cartesian product of the underlying sheaves of sets (c.f. (3a)) and the cartesian product of sheaves agrees with their cartesian product as presheaves, so for an open $U \subseteq X_1 \times_Z X_2$ we have

$$(\mathcal{O}_{X_1} \times_{\mathcal{O}_Z} \mathcal{O}_{X_2})(U) = \mathcal{O}_{X_1}(U \cap X_1) \times_{\mathcal{O}_Z(U \cap Z)} \mathcal{O}_{X_2}(U \cap X_2).$$

Hints: Certainly we can solve the lifting problem

$$\begin{array}{ccc} Z & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X_1 \amalg_Z X_2 \\ & \searrow g_2 & \downarrow g \\ & & Y \end{array}$$

(uniquely) on the level of topological spaces. By adjunction and the equality $gj_i = g_i$, to give a map $g^{-1}\mathcal{O}_Y \rightarrow \iota_{i*}\mathcal{O}_{X_i}$ is the same thing as giving a map $g_i^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_{X_i}$, which we have as part of the data of the ringed space map $g_i : X_i \rightarrow Y$. Commutativity of the diagram implies that the maps $g^{-1}\mathcal{O}_Y \rightarrow \iota_{i*}\mathcal{O}_{X_i}$ thus obtained agree on composing with the maps to the pushforward of \mathcal{O}_Z , hence they determine a map from $g^{-1}\mathcal{O}_Y$ to the structure sheaf of the pushout.

- (c) Show that the maps $X_i \rightarrow X_1 \times_Z X_2$ are closed embeddings of ringed spaces.
 (d) Show that there is an isomorphism of ringed spaces

$$(X_1 \amalg_Z X_2) \setminus Z \cong (X_1 \setminus Z) \amalg (X_2 \setminus Z).$$

- (e) If the diagram in (4b) is a diagram of closed embeddings of locally ringed spaces, then show that the pushout is also a locally ringed space, and the maps from X_i to it are morphisms of locally ringed spaces. Express the local ring of $X_1 \amalg_Z X_2$ at a point x in terms of those of X_1, X_2 , and Z ; the answer will depend on whether $x \in Z$.
 (f) Given a commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & W \end{array}$$

of closed embeddings of ringed spaces, show that it is a pushout iff for some (equivalently any) open cover $\{U_i \rightarrow W\}$ of W , the diagram

$$\begin{array}{ccc} (j_1 i_1)^{-1}U_i & \xrightarrow{i_1} & j_1^{-1}U_i \\ i_2 \downarrow & & \downarrow j_1 \\ j_2^{-1}U_i & \xrightarrow{j_2} & U_i \end{array}$$

is a pushout for each i . We say: “being a pushout is local on the hypothetical pushout”.

(g) Given a pushout diagram

$$\begin{array}{ccc} Z & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & X_1 \amalg_Z X_2 \end{array}$$

of ringed spaces and open covers $\{U_i^1 \rightarrow X_1\}$ and $\{U_j^2 \rightarrow X_2\}$, show that the pushouts

$$U_i^1 \amalg_{i_1^{-1}U_i^1 \cap i_2^{-1}U_j^2} U_j^2$$

form an open cover of $X_1 \amalg_Z X_2$.

(h) For closed embeddings $Z \rightarrow X_i$ of schemes, show that the ringed space pushout $X_1 \amalg_Z X_2$ is a scheme, so that, in particular, it is a pushout in the category of schemes. *Hint:* Being a scheme is local, so by the previous part, we reduce to the case where $X_1 = \text{Spec } B_1, X_2 = \text{Spec } B_2$ (hence also $Z = \text{Spec } C$) are affine, and we will try to show that the natural map

$$X_1 \amalg_Z X_2 \rightarrow \text{Spec}(B_1 \times_C B_2)$$

is an isomorphism. By (3g) it is a homeomorphism of spaces. To show that the map of sheaves of rings on this space is an isomorphism, it is enough to check it on the basis of opens of $\text{Spec}(B_1 \times_C B_2)$ given by localizations $\text{Spec}(B_1 \times_C B_2)_{(b_1, b_2)}$ at powers of a single element. For this, use (3d).

(i) If $\mathcal{I}_i \subseteq \mathcal{O}_{X_i}$ is the ideal of Z in X_i , then the ideal of Z in $X_1 \times_Z X_2$ is

$$\mathcal{I}_1 \times \mathcal{I}_2 \subseteq \mathcal{O}_{X_1} \times_{\mathcal{O}_Z} \mathcal{O}_{X_2}.$$

(5) **Gluing along closed subschemes III.**

- (a) Give an example to show that the functor Spec need not take a cartesian square in the category of rings to a cocartesian square in the category of schemes. That is, $\text{Spec} : \mathbf{An}^{\text{op}} \rightarrow \mathbf{Sch}$ is not right exact. *Hint:* The pushout of affine schemes need not be affine.
- (b) Show, however, that when f_1 and f_2 are surjective in the cartesian diagram

$$\begin{array}{ccc} B_1 \times_C B_2 & \xrightarrow{\pi_2} & B_2 \\ \pi_1 \downarrow & & \downarrow f_2 \\ B_1 & \xrightarrow{f_1} & C \end{array}$$

then Spec takes this diagram to a cartesian (c.f. (3e)) and cocartesian diagram of schemes.

(c) Let

$$\begin{array}{ccc} Z & \xrightarrow{i_1} & X_1 \\ i_2 \downarrow & & \downarrow j_1 \\ X_2 & \xrightarrow{j_2} & W \end{array}$$

be a commutative diagram of closed embeddings of schemes. If the diagram is a pushout diagram, and $W' \rightarrow W$ is flat, then prove that the diagram

$$\begin{array}{ccc} Z' & \xrightarrow{i'_1} & X'_1 \\ i'_2 \downarrow & & \downarrow j'_1 \\ X'_2 & \xrightarrow{j'_2} & W' \end{array}$$

obtained by base change is a pushout diagram. Conversely, if $W' \rightarrow W$ is an fppf cover and the pullback diagram is cocartesian, then show that the original diagram is cocartesian as well.

(6) **Gluing along closed subschemes IV: Producing Singularities.**

- (a) Observe that the “node” $\mathbb{Z}[x, y]/xy$ is isomorphic to the fibered product ring $\mathbb{Z}[x] \times_{\mathbb{Z}} \mathbb{Z}[y]$, where the maps to \mathbb{Z} kill x and y .
- (b) Show that two lines meeting to second order (“tangentially”) are given by the pushout of two lines over a point thickened to order 2. That is, show that the cocartesian diagram of rings

$$\begin{array}{ccc} \mathbb{Z}[x, y]/(y(y - x^2)) & \longrightarrow & \mathbb{Z}[x, y]/(y) \\ \downarrow & & \downarrow \\ \mathbb{Z}[x, y]/(y - x^2) & \longrightarrow & \mathbb{Z}[x, y]/(y, y - x^2) \end{array}$$

is also cartesian. Show that this diagram is isomorphic to

$$\begin{array}{ccc} \mathbb{Z}[x, y]/(y(y - x^2)) & \xrightarrow{f(x, y) \mapsto f(x, 0)} & \mathbb{Z}[x] \\ f(x, y) \mapsto f(x, x^2) \downarrow & & \downarrow \\ \mathbb{Z}[x] & \longrightarrow & \mathbb{Z}[x]/(x^2) \end{array}$$

so the claim is that the natural map

$$\begin{aligned} \mathbb{Z}[x, y]/(y(y - x^2)) &\rightarrow \mathbb{Z}[x] \times_{\mathbb{Z}[x]/(x^2)} \mathbb{Z}[x] \\ f(x, y) &\mapsto (f(x, x^2), f(x, 0)) \end{aligned}$$

is an isomorphism of rings.

- (c) Observe that the cusp $\mathbb{Z}[x, y]/(y^2 - x^3)$ is isomorphic to the monoid algebra on the submonoid $P \subseteq \mathbb{N}$ generated by 2, 3. Show that

$$\mathbb{Z}[P] \hookrightarrow \mathbb{Z}[\mathbb{N}] \rightrightarrows \mathbb{Z}[x]/x^2$$

is an equalizer diagram of rings (equivalently sets), where the two maps from $\mathbb{Z}[\mathbb{N}] \cong \mathbb{Z}[x]$ are given by $x \mapsto 0$ and $x \mapsto x$.

- (d) Let C be a smooth curve over a field k and let x be a closed point of C with residue field $K = k(x) = \mathcal{O}_C/\mathfrak{m}_x$. Suppose K/k is a finite separable field extension. Show that there is a unique k algebra section s of $\mathcal{O}_C/\mathfrak{m}_x^2 \rightarrow K$. *Hint:* This is a square zero k algebra extension of K by $\mathfrak{m}_x/\mathfrak{m}_x^2$ so sections of it are a pseudo-torsor under $\text{Hom}_K(\Omega_{K/k}, \mathfrak{m}_x/\mathfrak{m}_x^2)$.
- (e) Consider the diagram

$$\mathcal{O}_C \rightrightarrows \mathcal{O}_C/\mathfrak{m}_x^2$$

of sheaves of rings on C where one map is the natural projection, and the other map is the composition of the natural projection $\mathcal{O}_C \rightarrow \mathcal{O}_C/\mathfrak{m}_x = K$ and the section s . Let $\mathcal{O}_{C'}$ denote the equalizer ring (sheaf of rings) of this diagram. Show that the ringed space $C' := (\text{sp } C, \mathcal{O}_{C'})$ is a scheme with a cusp singularity at the point $x \in \text{sp } C$; Spec of the inclusion $\mathcal{O}_{C'} \hookrightarrow \mathcal{O}_C$ is a map of schemes $C \rightarrow C'$ which gives the normalization of this cusp singularity.

- (f) An alternative geometric treatment of the previous part goes as follows. Consider a diagram of schemes $\text{Spec } K[\epsilon]/\epsilon^2 \rightrightarrows C$ where one map corresponds to $0 \in T_x C$ (i.e. factors through $\text{Spec}(K \hookrightarrow K[\epsilon]/\epsilon^2)$) and the other map corresponds to a nonzero element $v \in T_x C \cong K$ under the usual description of maps out of Spec of the dual numbers (Hartshorne II.2.8, or Exercise 18 below). Show that the direct limit C' of this diagram (taken in ringed spaces if you want) is a scheme (so it is a fortiori the direct limit in schemes as well) and in fact $C \rightarrow C'$ is an isomorphism on topological spaces (this is obvious because the two maps out of $\text{Spec } K[\epsilon]/\epsilon^2$ already agree on the level of topological spaces) and C' is a curve with a cusp singularity at x . Notice that the choice of v is rather irrelevant: two different choices would yield two maps $\text{Spec } K[\epsilon]/\epsilon^2 \rightarrow C$ which differ by a K automorphism of $K[\epsilon]/\epsilon^2$, so the direct limits will be isomorphic (as schemes under C).
- (g) Rephrase parts (6a) and (6b) in similar geometric language to show how one produces a nodal curve or a “tangentially glued” curve by appropriately pushing out.
- (7) **More On Singularity Production.** We saw in the previous exercise how to produce singularities by taking an appropriate equalizer or inverse limit of rings. In general, one can produce more elaborate singularities by finding appropriate subrings that may not have simple descriptions in terms of inverse limits. Consider, for example, the *ramphoid cusp* curve singularity $\mathbb{Z}[x, y]/(y^2 - x^5)$. This ring is the monoid algebra on the submonoid $R \subseteq \mathbb{N}$ generated by 2, 5 and the inclusion $R \hookrightarrow \mathbb{N}$ induces a map $\mathbb{Z}[R] \hookrightarrow \mathbb{Z}[\mathbb{N}]$ which is the normalization of the ramphoid cusp.

Let C be a smooth curve over a field k and let $x \in C(k)$ be a k point of C . Suppose we want to introduce a ramphoid cusp singularity in C at x . That is, we want a map $C \rightarrow C'$ of curves over k which is bijective, C' should have a ramphoid cusp singularity at x , and the map should be an isomorphism away from x . Unlike the introduction of a cusp singularity, this will require some additional choices (so-called *moduli of crimping data*). Let us concentrate on some affine open $\text{Spec } A \subseteq C$ containing x , with A a PID under k . Choose a uniformizer $t \in \mathfrak{m}_x \subseteq A$ (i.e. an element of \mathfrak{m}_x not in \mathfrak{m}_x^2 , hence serving as a basis of $\mathfrak{m}_x/\mathfrak{m}_x^2 \cong k$). Let $A(t) \subseteq A$ be the subring consisting of those $a \in A$ such that

$$\bar{a} = a_0 + a_2 t^2 \in A/\mathfrak{m}_x^4 \quad \text{for some } a_0, a_2 \in k.$$

Note $t^2, t^4, t^5, t^6, \dots \in A(t)$, but $t, t^3 \notin A(t)$ since $A/\mathfrak{m}_x^4 \cong k[t]/t^4$.

- (a) Assume $\text{char } k \neq 2$. If t' is another uniformizer, then the subrings $A(t) \subseteq A$ and $A(t') \subseteq A$ coincide iff $[t] = [t'] \in (\mathfrak{m}_x/\mathfrak{m}_x^3)/k^*$.
- (b) The assumption $\text{char } k \neq 2$ cannot be removed: in $\mathbb{F}_2[z]$, $t = z$ and $t' = z + z^2$ are both uniformizers at the origin, and $[t] \neq [t'] \in \mathbb{F}_2[z]/z^4$, but nevertheless, $\mathbb{F}_2[z](t) = \mathbb{F}_2[z](t') = \mathbb{F}_2[z^2, z^5]$.
- (c) The set of $[t] \in (\mathfrak{m}_x/\mathfrak{m}_x^3)/k^*$ with $t \notin \mathfrak{m}_x^2$ is a torsor under k .

- (d) The inclusion of rings $A(t) \hookrightarrow A$ induces a homeomorphism of topological spaces

$$\begin{aligned} \text{Spec } A &\rightarrow \text{Spec } A(t) \\ \wp &\mapsto \wp \cap A(t) \end{aligned}$$

taking the maximal ideal $\mathfrak{m}_x = (t)$ of A to the maximal ideal $\overline{\mathfrak{m}}_x = (t) \cap A(t) = (t^2, t^5)$ of $A(t)$ and an isomorphism $A(t)_{t^2, t^5} \cong A_t$ “away from x ”.

- (e) The commutative diagram

$$\begin{array}{ccc} k[z^2, z^5] & \longrightarrow & A(t) \\ \downarrow & & \downarrow \\ k[z] & \xrightarrow{z \mapsto t} & A \end{array}$$

of rings is both cartesian and cocartesian and in the corresponding diagram of local rings

$$\begin{array}{ccc} k[z^2, z^5]_{(z^2, z^5)} & \hookrightarrow & A(t)_{\overline{\mathfrak{m}}_x} \\ \downarrow & & \downarrow \\ k[z]_{(z)} & \xrightarrow{z \mapsto t} & A_x \end{array}$$

both horizontal arrows are étale maps of local rings.

- (f) Conversely, if $B \subseteq A$ is a subring inducing a bijection on prime ideals, an isomorphism away from x , and a cartesian and cocartesian diagram

$$\begin{array}{ccc} k[z^2, z^5]_{(z^2, z^5)} & \hookrightarrow & B_{\overline{\mathfrak{m}}_x} \\ \downarrow & & \downarrow \\ k[z]_{(z)} & \xrightarrow{z \mapsto t} & A_x \end{array}$$

of local rings, then show that $B = A(t)$.

From the above, it is clear that we can replace $\text{Spec } A \subseteq C$ with $\text{Spec } A(t)$ to form a new curve C' , and that $A(t) \hookrightarrow A$ induces a map $C \rightarrow C'$ with the desired properties. Notice that the set of maps $C \rightarrow C'$ with the desired properties is a torsor under k .

- (8) Let A be a ring.
- Show that the symmetric algebra functor Sym_A^* is left adjoint to the forgetful functor from A -algebras to A -modules.
 - Show that Sym_A^* takes finitely generated modules to finite-type A -algebras and finitely presented A -modules to finitely presented A -algebras.
 - Sym_A^* takes (finite) direct sums to the corresponding tensor products.
 - For any A -module M , show that the map $d : \text{Sym}_A^* M \rightarrow (\text{Sym}_A^* M) \otimes_A M$ given by

$$m_1 \cdots m_n \mapsto \sum_{i=1}^n m_1 \cdots \hat{m}_i \cdots m \otimes m_i$$

is an A -linear derivation, hence by the universal property of $\Omega_{\text{Sym}_A^* M/A}$ it determines a map

$$\Omega_{\text{Sym}_A^* M/A} \rightarrow (\text{Sym}_A^* M) \otimes_A M$$

of $\mathrm{Sym}_A^* M$ -modules.

- (e) Show that this map is an isomorphism. c.f. Illusie 1.1.3.2.
 (f) For any A -module M , using Sym_A^* of the module maps

$$\begin{aligned} \Delta : M &\rightarrow M \oplus M \\ \cdot(-1) : M &\rightarrow M \\ 0 : M &\rightarrow A \end{aligned}$$

(with the identification $\mathrm{Sym}_A^*(M \oplus M) = \mathrm{Sym}_A^* M \otimes_A \mathrm{Sym}_A^* M$) as comultiplication, coinverse, and counit maps respectively endows $\mathrm{Sym}_A^* M$ with the structure of a cogroup object in the category of A -algebras. Taking Spec we get a group object in the category of affine schemes over $\mathrm{Spec} A$.

- (g) The functorial nature of these constructions ensures that they sheafify: for any site X and sheaf of rings A on X , there is a left adjoint to the forgetful functor from A -modules to A -algebras obtained by first constructing the adjoint at the level of presheaves, then using the adjointness property of the sheafification functor. The appropriate invariance of existence of adjoints under composition of equivalences of categories hence ensures that analogous adjoints exist for the category of module and algebra objects over a ring object of a topos.
- (9) Let $X = \mathrm{Spec} A$ be an affine scheme, \underline{A} the constant sheaf of rings on $\mathrm{Spec} A$.
- (a) The natural map $\underline{A} \rightarrow \mathcal{O}_X$ is flat and is an epimorphism in the category of sheaves of rings. Illusie refers to a flat surjective morphism of ring objects of a topos as a *localization morphism*.
- (b) For any A -module M , the (quasi-coherent) \mathcal{O}_X -modules M^\sim and $\underline{M} \otimes_{\underline{A}} \mathcal{O}_X$ coincide.
- (10) Prove that the infinite plane with the origin doubled is not quasi-separated.
- (11) Show that the sheaf of continuous real-valued functions on the real line is not a coherent module over itself (here “coherent” is in the sense of [EGA], so an \mathcal{O}_X -module \mathcal{F} is *coherent* iff it is finitely generated, and for any open $U \subseteq X$ and any \mathcal{O}_X -module map $f : \mathcal{O}_X|_U^n \rightarrow \mathcal{F}|_U$, the kernel of f is finitely generated). Of course, *finitely generated* means that locally there is a surjection $\mathcal{O}_X^n \rightarrow \mathcal{F}$.
- (12) Construct a scheme with no closed points.
- (13) Show that any scheme whose underlying topological space has at most two points must be affine. Construct a scheme whose underlying space has three points which is not affine.
- (14) Give an example of a morphism of schemes which is an epimorphism on the underlying topological spaces, but which fails to be an epimorphism on topological spaces after a base change.
- (15) Give an example of a morphism of schemes which is surjective on topological spaces, but not a categorical epimorphism. *Hint*: You only need one point.
- (16) Recall that a morphism of topological spaces $f : X \rightarrow Y$ is *quasi-compact* if, for every quasi-compact subspace $A \subseteq X$, the preimage $f^{-1}[A]$ is quasi-compact. A morphism of schemes is quasi-compact if the underlying morphism of topological spaces is quasi-compact.
- (a) A morphism of schemes $f : X \rightarrow Y$ is quasi-compact if and only if, for any open affine $U \subseteq Y$, the preimage $f^{-1}[U]$ has a Zariski cover by an affine scheme.
- (b) Show that quasi-compact morphisms are preserved under base change.

- (c) Show that a flat, quasi-compact morphism of schemes which is an epimorphism on topological spaces (an *fppc cover*) is also categorical epimorphism and retains all of these properties after arbitrary base change.
- (17) Let $f : X \rightarrow Y$ be a finitely presented (locally finitely presented and quasi-compact) morphism of schemes. Show that the restriction of f to an open subscheme of X is finitely presented.
- (18) For a scheme X , write $X[\epsilon]$ as shorthand for $X \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Z}[\epsilon]/\langle \epsilon^2 \rangle$. Let k be a field, X a scheme over $\mathrm{Spec} k$.
- (a) Show that the (contravariant) functor $Y \mapsto \mathrm{Hom}_{\mathbf{Sch}/k}(Y[\epsilon], X)$ is represented by

$$TX := \mathrm{Spec} \mathrm{Sym}_{\mathcal{O}_X}^* \Omega_{X/k}.$$

Taking $Y = \mathrm{Spec} k$ this recovers a Hartshorne exercise. Another way to say this is that $X \mapsto TX$ is right adjoint to the base change $Y \mapsto Y[\epsilon]$, which is to say that TX is the Weil restriction of X along $\mathbb{Z} \rightarrow \mathbb{Z}[\epsilon]/\epsilon^2$. In general, there is also a Weil restriction $J^n X$ of X along $\mathbb{Z} \rightarrow \mathbb{Z}[\epsilon]/\epsilon^{n+1}$, called the n^{th} jet scheme of X .

- (b) If we fix a morphism $Y \rightarrow X$ and view Y as an object of the category of schemes over X , then we have a natural bijection

$$\mathrm{Hom}_{\mathbf{Sch}/X}(Y, TX) = \mathrm{Hom}_{Y \setminus \mathbf{Sch}}(Y[\epsilon], X).$$

Here $Y \setminus \mathbf{Sch}$ is the category of schemes under Y .

- (c) The scheme TX carries the structure of a group object in the category of schemes over X given by “addition in the fibers” (see Exercise 8). This endows the set on the left hand side of the above bijection with the structure of a group.
- (d) The scheme $Y[\epsilon]$ carries the structure of a cogroup object in the category of schemes under Y , such that the above bijection is an isomorphism of groups when the RHS has the group structure induced from the cogroup object structure of $Y[\epsilon]$.
- (19) Resolve the tac-node singularity $y^2 = x^4$ in one blow-up by blowing up along a carefully chosen ideal. *Hint:* The corresponding closed subscheme will not be reduced.
- (20) **Sheaf theory I.** Suppose $f : X \rightarrow Y$ is a continuous map of topological spaces which, locally on X , is a homeomorphism onto its image. Show that the adjunction morphism $f^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism for any sheaf \mathcal{F} on X . *Hint:* It is easier to prove the stronger statement that $f_{\mathrm{pre}}^{-1}f_*\mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism of presheaves.
- (21) **Sheaf theory II.** Let $f : X \rightarrow Y$ be a continuous map of topological spaces, \mathcal{F} a sheaf on Y .
- (a) Show that the natural map

$$\varinjlim_U \mathcal{F}(U) \rightarrow \Gamma(X, f^{-1}\mathcal{F})$$

is monic (the direct limit is over open neighborhoods U of $f[X]$ in Y).

- (b) If Y is paracompact and $f[X]$ is closed in Y , then show that it is an isomorphism.
- (c) Construct an example where $f : X \rightarrow Y$ is a closed embedding of finite topological spaces, but this map is not surjective.

- (d) In the next parts, we will give an example showing that “paracompact” cannot be replaced with “completely regular”. Let S be the Sorgenfrey line (\mathbb{R} with basic open sets $[a, b)$ for $a < b$), let $Y = S \times S$, and let $f : X \rightarrow Y$ be the inclusion of the skew diagonal $\{(-x, x) : x \in \mathbb{R}\}$ in the subspace topology it inherits from Y . Let \mathcal{F} be the constant sheaf associated to the two element set $\{0, 1\}$. Show that X has the discrete topology, hence

$$\begin{aligned} \Gamma(X, f^{-1}\mathcal{F}) &= \prod_{x \in X} \mathcal{F}_x \\ &= \{0, 1\}^{\mathbb{R}}. \end{aligned}$$

- (e) Show that the characteristic function of the irrational numbers $f \in \{0, 1\}^{\mathbb{R}}$ is not in the image of the natural map. *Hint:* Baire category theorem.

- (22) **Sheaf theory III.** In this problem we will be entirely concerned with sheaves on the Sierpinski space $X = \{0, 1\}$ with open sets $\{\emptyset, \{1\}, X\}$. Note that $\{0\}$ is closed in X but $\{1\}^- = X$. Let \mathbf{A} be the (abelian) category of sheaves of abelian groups on X .

- (a) For any $\mathcal{F} \in \mathbf{A}$, show that $\mathcal{F}(X) = \mathcal{F}_0$ and $\mathcal{F}(\{1\}) = \mathcal{F}_1$ so that the restriction map $\mathcal{F}(X) \rightarrow \mathcal{F}(\{1\})$ may be regarded as a map $f : \mathcal{F}_0 \rightarrow \mathcal{F}_1$. Note that $\mathcal{F} \mapsto (\mathcal{F}_0 \rightarrow \mathcal{F}_1)$ is an equivalence of categories between \mathbf{A} and the category of arrows in \mathbf{Ab} .
- (b) Note that $f : \mathcal{F}_0 \rightarrow \mathcal{F}_1$ is flasque iff f is surjective. Prove that \mathcal{F} is injective in \mathbf{A} iff $f : \mathcal{F}_0 \rightarrow \mathcal{F}_1$ is a split epimorphism of injective abelian groups. That is, every injective in \mathbf{A} is of the form $\pi_1 : I \oplus J \rightarrow I$ for divisible abelian groups I, J .
- (c) Show that the global section functor $\Gamma : \mathbf{A} \rightarrow \mathbf{Ab}$ is exact.
- (d) Let Γ_0 be the subfunctor of Γ given by sections with support in $\{0\}$. Show that $\Gamma_0(f : \mathcal{F}_0 \rightarrow \mathcal{F}_1) = \text{Ker } f$.
- (e) Show that $R^1\Gamma_0(f : \mathcal{F}_0 \rightarrow \mathcal{F}_1) = \text{Cok } f$ and $R^{>1}\Gamma_0 = 0$. In fact, show that the δ -functor $R^\bullet\Gamma_0$ is “identified with the Snake Lemma sequence”. That is, for a short exact sequence

$$\underline{\mathcal{F}} = 0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$$

on X , the associated long exact sequence of $R^i\Gamma_0$ is identified (functorially in $\underline{\mathcal{F}}$) with the exact sequence

$$0 \rightarrow \text{Ker } f' \rightarrow \text{Ker } f \rightarrow \text{Ker } f'' \rightarrow \text{Cok } f' \rightarrow \text{Cok } f \rightarrow \text{Cok } f'' \rightarrow 0$$

obtained from the Snake Lemma. *Hint:* One way to do this is to prove that “the Snake Lemma” is an effaceable δ -functor from \mathbf{A} to \mathbf{Ab} , whose degree zero part agrees with Γ_0 by (22d).

- (23) **Sheaf theory IV.** We continue the previous exercise by studying Poincaré duality on the Sierpinski space $X = \{0, 1\}$ (with open sets $\{\emptyset, \{1\}, X\}$) for the functor Γ_0 given by global sections with support in $\{0\}$. For simplicity, we work over a field k . Let $\underline{k} = (\text{Id} : k \rightarrow k)$ be the constant sheaf on X and let $\mathbf{Mod}(\underline{k})$ be the category of \underline{k} modules. Let $D_c^b(\underline{k})$ be the bounded derived category of \underline{k} modules with finitely generated cohomology sheaves, and let $D_c^b(k)$ be the analogous category of k vector spaces.

- (a) Show that $\underline{k} \in \mathbf{Mod}(\underline{k})$ is both injective and projective, that $P := (0 \rightarrow k) \in \mathbf{Mod}(\underline{k})$ is projective, and that $I := (k \rightarrow 0) \in \mathbf{Mod}(\underline{k})$ is injective.

(b) Show that any finitely generated $\mathcal{F} = (\mathcal{F}_0 \rightarrow \mathcal{F}_1) \in \mathbf{Mod}(\underline{k})$ is isomorphic to a finite direct sum of copies of \underline{k} , P , and I .

(c) Show that

$$\begin{aligned} \Gamma_0^! : D_c^b(k) &\rightarrow D_c^b(\underline{k}) \\ V^\bullet &\mapsto (0 \rightarrow V)^\bullet[1] \end{aligned}$$

is right adjoint to $\mathbf{R}\Gamma_0 : D_c^b(\underline{k}) \rightarrow D_c^b(k)$. In fact, there is a $D_c^b(k)$ isomorphism

$$\mathbf{R}\mathrm{Hom}_k^\bullet(\mathbf{R}\Gamma_0 \mathcal{F}^\bullet, V^\bullet) = \mathbf{R}\mathrm{Hom}_{\underline{k}}^\bullet(\mathcal{F}^\bullet, \Gamma_0^! V^\bullet)$$

natural in \mathcal{F}^\bullet and V^\bullet .

Hint: Reduce to the case where \mathcal{F}^\bullet is \underline{k} , P , or I and $V^\bullet = k$.

(24) Let $G = \mathrm{Spec} B$ be a group object in the category \mathcal{C} of affine schemes over $\mathrm{Spec} A$ (a Hopf A -algebra), so the structure maps

$$m : G \times G \rightarrow G, \quad I : G \rightarrow G, \quad e : \mathrm{Spec} A \rightarrow G$$

are given by A -algebra morphisms

$$m^\# : B \rightarrow B \otimes_A B, \quad I^\# : B \rightarrow B, \quad e^\# : B \rightarrow A.$$

Suppose that Spec of the A -algebra morphism

$$a^\# : C \rightarrow B \otimes_A C$$

defines an action $a : G \times \mathrm{Spec} C \rightarrow \mathrm{Spec} C$. Show that the set

$$C^G := \{c \in C : a(c) = 1 \otimes c\}$$

is a sub- A -algebra of C and define $(\mathrm{Spec} C)/G := \mathrm{Spec} C^G$. Show that the map $\mathrm{Spec} C \rightarrow (\mathrm{Spec} C)/G$ induced by the inclusion $C^G \hookrightarrow C$ is a *categorical quotient*:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}((\mathrm{Spec} C)/G, \mathrm{Spec} D) &= \{f \in \mathrm{Hom}_{\mathcal{C}}(\mathrm{Spec} C, \mathrm{Spec} D) : af = \pi_2 f\} \\ &= \{f^\# \in \mathrm{Hom}_{A\text{-alg}}(D, C) : a^\# f^\# = 1 \otimes f^\#\} \end{aligned}$$

In other words,

$$G \times \mathrm{Spec} C \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{\pi_2} \end{array} \mathrm{Spec} C \longrightarrow \mathrm{Spec} C^G$$

is a coequalizer diagram in \mathcal{C} or (equivalently)

$$C^G \hookrightarrow C \begin{array}{c} \xrightarrow{a^\#} \\ \xrightarrow{1 \otimes \mathrm{Id}_C} \end{array} B \otimes_A C$$

is an equalizer diagram in the category of A -algebras.

(25) **Dualizing sheaf of a prestable curve.** Let $C \rightarrow S$ be a nodal curve. (You can assume $S = \mathrm{Spec} \mathbb{C}$ if you wish, in which case “nodal” means “proper” and étale locally $\mathrm{Spec} \mathbb{C}[x]$ or $\mathrm{Spec} \mathbb{C}[x, y]/xy$.) Show that the dualizing sheaf ω_C of C is locally free and describe it in terms of the normalization $\eta : \overline{C} \rightarrow C$. Show that the sheaf of log differentials $\Omega_{C/S}^\dagger$ is a dualizing sheaf for C/S .

(26) **Differentiable spaces.** Let n be a nonnegative integer, U an open subset of \mathbb{R}^n , f_1, \dots, f_k smooth real valued functions on U . Let $\mathcal{O}_U = \mathcal{O}_{\mathbb{R}^n}|_U$ be the sheaf of \mathbb{R} -algebras given by smooth real valued functions, so $f_i \in \Gamma(U, \mathcal{O}_U)$. Let $i : Z \hookrightarrow U$ be their common zero locus. Endow Z with the sheaf of \mathbb{R} -algebras

$$\mathcal{O}_Z := i^{-1}(\mathcal{O}_U / \langle f_1, \dots, f_k \rangle) = (i^{-1} \mathcal{O}_U) / \langle f_1|_U, \dots, f_k|_U \rangle.$$

Note that (Z, \mathcal{O}_Z) is a locally ringed space over \mathbb{R} (i.e. over the one point space with sheaf of rings \mathbb{R}) with $\mathcal{O}_{Z,z}/\mathfrak{m}_z \cong \mathbb{R}$ for every point $z \in Z$. A *differentiable space* is a locally ringed space over \mathbb{R} (with paracompact topological space, say) locally isomorphic (as a locally ringed space over \mathbb{R}) to one of these “model” (Z, \mathcal{O}_Z) . Differentiable spaces form a full subcategory of the category of locally ringed spaces over \mathbb{R} .

- (a) Show that a morphism of differential spaces is what you think it should be. More precisely, let $(U \subset \mathbb{R}^n, f_1, \dots, f_k, Z)$ and $(V \subset \mathbb{R}^m, g_1, \dots, g_l, W)$ be the data specifying two “model” differential spaces. Suppose $F : U' \rightarrow V$ is a smooth function defined on an open neighborhood U' of Z in U such that

$$g_1 F, \dots, g_l F \in \langle f_1, \dots, f_k \rangle|_{U'}.$$

Then show that $F(Z) \subset W$ and that F determines a morphism of differential spaces $Z \rightarrow W$ in a natural way. Show that every morphism of differential spaces $(Z, \mathcal{O}_Z) \rightarrow (W, \mathcal{O}_W)$ arises in this manner.

- (b) As with any map of ringed spaces, $f : X \rightarrow Y$ is a *closed embedding* if the map on topological spaces is a closed embedding and $\mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$ is surjective; we refer to the kernel I of this surjection as the ideal of the closed embedding. If $I^2 = 0$, then the closed embedding is called *square zero*; in this case I has a natural \mathcal{O}_X module structure characterized by $f \cdot i = g \cdot i$ for any $g \in f^{-1} \mathcal{O}_Y$ mapping to $f \in \mathcal{O}_X$. Show that a square zero closed embedding of differential spaces is a homeomorphism on topological spaces.
- (c) A differential space is called *smooth* if it is locally isomorphic to \mathbb{R}^n (with its usual sheaf of smooth functions). A differential space X is called *formally smooth* if, for any square zero closed embedding $Y \hookrightarrow Y'$ of differential spaces and any morphism $f : Y \rightarrow X$, there is a morphism $f' : Y' \rightarrow X$ restricting to f on Y . Prove or disprove: A differential space is smooth iff it is formally smooth.

- (27) Let $\mu_n = \text{Spec } \mathbb{Z}[t]/\langle t^n - 1 \rangle$ be the affine group scheme whose structure maps

$$m : \mu_n \times \mu_n \rightarrow \mu_n, \quad I : \mu_n \rightarrow \mu_n, \quad e : \text{Spec } \mathbb{Z} \rightarrow \mu_n$$

are given by Spec of the ring maps:

$$\begin{aligned} \frac{\mathbb{Z}[t]}{\langle t^n - 1 \rangle} &\rightarrow \frac{\mathbb{Z}[t]}{\langle t^n - 1 \rangle} \otimes \frac{\mathbb{Z}[t]}{\langle t^n - 1 \rangle} \\ t &\mapsto t \otimes t \end{aligned}$$

$$\begin{aligned} \frac{\mathbb{Z}[t]}{\langle t^n - 1 \rangle} &\rightarrow \frac{\mathbb{Z}[t]}{\langle t^n - 1 \rangle} \\ t &\mapsto t^{n-1} \end{aligned}$$

$$\begin{aligned} \frac{\mathbb{Z}[t]}{\langle t^n - 1 \rangle} &\rightarrow \mathbb{Z} \\ t &\mapsto 1 \end{aligned}$$

and let \mathbb{G}_m be the affine group scheme $\text{Spec } \mathbb{Z}[t, t^{-1}]$ with multiplication map $t \mapsto t \otimes t$.

- (a) Show that μ_n is finite (hence proper) over $\text{Spec } \mathbb{Z}$.
- (b) Let X be a scheme such that n is invertible in each ring $\mathcal{O}_X(U)$ (for example, any scheme over a field of characteristic zero). Then $\mu_n(X) := X \times_{\text{Spec } \mathbb{Z}} \mu_n$ is finite étale over X . For example, $\text{Spec } \mathbb{C} = \mu_2(\text{Spec } \mathbb{R})$ is étale over $\text{Spec } \mathbb{R}$.

(c) Over any splitting field k for $x^n - 1$ of characteristic p not dividing n we have

$$\mu_n(\text{Spec } k) \cong \coprod_{\mathbb{Z}/n\mathbb{Z}} \text{Spec } k \cong \text{Spec} \left(\bigoplus_{g \in \mathbb{Z}/n\mathbb{Z}} \text{Spec } k \right)$$

(as group schemes).

(d) Show that giving an action of \mathbb{G}_m on $\text{Spec } A$ (a ring map $a : A \rightarrow A[t, t^{-1}]$ respecting multiplication appropriately) is the same thing as giving A a \mathbb{Z} -grading. Show that giving a μ_n action on $\text{Spec } A$ is the same thing as giving A a $\mathbb{Z}/n\mathbb{Z}$ -grading. In either case, the fixed locus is the degree zero subring. *Hint:* Given the action a , define the grading by

$$A_n := \{x \in A : a(x) = xt^n\}$$

and given the grading $A = \bigoplus_{n \in \mathbb{Z}} A_n$, define the action by

$$a\left(\sum_i a_i\right) := \sum_i a_i t^i.$$

(28) Taking Spec of the ring map

$$\begin{array}{ccc} \mathbb{Z}[x] & \rightarrow & \mathbb{Z}[t]/\langle t^n - 1 \rangle \otimes_{\mathbb{Z}} \mathbb{Z}[x] = \mathbb{Z}[t, x]/\langle t^n - 1 \rangle \\ x & \mapsto & tx \end{array}$$

defines an action $a : \mu_n \times \mathbb{A}^1 \rightarrow \mathbb{A}^1$. Taking Spec of the inclusion of rings $\mathbb{Z}[x^n] \hookrightarrow \mathbb{Z}[x]$ gives a quotient map $\mathbb{A}^1 \rightarrow \mathbb{A}^1/\mu_n \cong \mathbb{A}^1$ for this action. Set $y = x^n$. Then, as algebras over $\mathbb{Z}[y]$, there is an isomorphism $\mathbb{Z}[x] \cong \mathbb{Z}[y][w]/\langle w^n - y \rangle$. The étale locus of the quotient map is the Zariski open subset where y and n are invertible.

(29) Define an action $a : \mu_n \times \mathbb{A}^2 \rightarrow \mathbb{A}^2$ by taking Spec of

$$\begin{array}{ccc} a^\sharp : \mathbb{Z}[x, y] & \rightarrow & \mathbb{Z}[t]/\langle t^n - 1 \rangle \otimes \mathbb{Z}[x, y] \\ x & \mapsto & t \otimes x \\ y & \mapsto & t^{n-1} \otimes y. \end{array}$$

Show that (Spec of) the map

$$\begin{array}{ccc} \frac{\mathbb{Z}[u, v, w]}{\langle uv - w^n \rangle} & \rightarrow & \mathbb{Z}[x, y] \\ u & \mapsto & x^n \\ v & \mapsto & y^n \\ w & \mapsto & xy \end{array}$$

gives an isomorphism onto the quotient $\mathbb{A}^2/\mu_n = \text{Spec } \mathbb{Z}[x, y]^{\mu_n}$ of this action (see Exercise 24).

(30) In the setup of the previous exercise, let $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2/\mu_2$ be the quotient map.

(a) Blowing up the origin (the closed subscheme with ideal $I := \langle u, v, w \rangle$) in \mathbb{A}^2/μ_2 gives a resolution of singularities $Y \rightarrow \mathbb{A}^2/\mathbb{Z}_2$.

(b) Y is isomorphic to the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$.

(c) The inverse image ideal sheaf $f^{-1}I$ is the square of the ideal of the origin in \mathbb{A}^2 and consequently we have

$$\text{Bl}_{f^{-1}I} \mathbb{A}^2 = \text{Bl}_{\langle x, y \rangle^2} \mathbb{A}^2 = \text{Bl}_{\langle x, y \rangle} \mathbb{A}^2$$

is the blow-up of the origin in \mathbb{A}^2 , which is (the total space of) $\mathcal{O}_{\mathbb{P}^1}(-1)$. (See Hartshorne Exercise 7.11(a) for one of the above equalities.)

(d) In the diagram

$$\begin{array}{ccc} \mathrm{Tot}\mathcal{O}(-1) \xrightarrow{g} W := \mathbb{A}^2 \times_{\mathbb{A}^2/\mu_2} \mathrm{Tot}\mathcal{O}(-2) & \longrightarrow & \mathrm{Tot}\mathcal{O}(-2) = Y \\ & & \downarrow \\ & & \mathbb{A}^2/\mu_2 \\ \downarrow & \xrightarrow{f} & \downarrow \\ \mathbb{A}^2 & & \mathbb{A}^2/\mu_2 \end{array}$$

of Exercise 1c, the closed embedding g gives an isomorphism on the underlying topological spaces, but the generic point of the zero-section is an embedded point in W .

(31) Let $U(1) := \mathrm{Spec}\mathbb{Z}[x, y]/\langle x^2 + y^2 - 1 \rangle$ be the group scheme over \mathbb{Z} with (co-)multiplication map

$$\begin{aligned} x &\mapsto x \otimes x - y \otimes y \\ y &\mapsto x \otimes y + y \otimes x \end{aligned}$$

and identity map

$$\begin{aligned} \mathbb{Z}[x, y]/\langle x^2 + y^2 - 1 \rangle &\rightarrow \mathbb{Z} \\ x &\mapsto 1 \\ y &\mapsto 0. \end{aligned}$$

Define a μ_2 action on the product group scheme $U(1) \times \mathbb{G}_m$ by giving x, y, t grading $1 \in \mathbb{Z}/2\mathbb{Z}$ (c.f. (27d)), and note that this action respects the group structure so that the fixed subscheme

$$U(1) \times_{\mu_2} \mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[x, y, t, t^{-1}]/\langle x^2 + y^2 - 1 \rangle)_0$$

is a group scheme.

- What is the inverse map for $U(1)$?
- Show that, after base changing along $\mathbb{Z} \rightarrow \mathbb{Z}[\sqrt{-1}]$, there is an isomorphism of group schemes $\mathbb{G}_m \cong U(1)$, but there is no such isomorphism (even just as schemes) of $U(1)$ and \mathbb{G}_m .
- Show that

$$\begin{aligned} (\mathbb{Z}[x, y, t, t^{-1}]/\langle x^2 + y^2 - 1 \rangle)_0 &\rightarrow \mathbb{Z}[x, y]_{x^2+y^2} \\ ty &\mapsto y \\ tx &\mapsto x \\ x^2 &\mapsto \frac{x^2}{x^2 + y^2} \\ y^2 &\mapsto \frac{y^2}{x^2 + y^2} \\ t^2 &\mapsto x^2 + y^2 \end{aligned}$$

defines an isomorphism of group schemes from $\mathbb{Z}[x, y]_{x^2+y^2}$ to $U(1) \times_{\mu_2} \mathbb{G}_m$ with inverse

$$\begin{aligned} x &\mapsto tx \\ y &\mapsto ty \end{aligned}$$

when $\mathbb{Z}[x, y]_{x^2+y^2}$ is given the multiplication, identity, and inverse maps

$$\begin{aligned} \mathbb{Z}[x, y]_{x^2+y^2} &\rightarrow \mathbb{Z}[x, y]_{x^2+y^2} \otimes \mathbb{Z}[x, y]_{x^2+y^2} \\ x &\mapsto x \otimes x - y \otimes y \\ y &\mapsto x \otimes y + y \otimes x \\ \mathbb{Z}[x, y]_{x^2+y^2} &\rightarrow \mathbb{Z}[x, y]_{x^2+y^2} \\ x &\mapsto \frac{x}{x^2+y^2} \\ y &\mapsto \frac{-y}{x^2+y^2} \\ \mathbb{Z}[x, y]_{x^2+y^2} &\rightarrow \mathbb{Z} \\ x &\mapsto 1 \\ y &\mapsto 0. \end{aligned}$$

(d) Let $A := \mathbb{Z}[x_1, x_2, y_1, y_2]/\langle x_1x_2 - y_1y_2 - 1, x_1y_2 + x_2y_1 \rangle$. View $\text{Spec } A$ as a group scheme with multiplication, inverse, and identity

$$\begin{aligned} A &\rightarrow A \otimes_{\mathbb{Z}} A \\ x_i &\mapsto x_i \otimes x_i - y_i \otimes y_i \\ y_i &\mapsto x_i \otimes y_i + y_i \otimes x_i \\ A &\rightarrow A \\ x_1 &\mapsto x_2 \\ x_2 &\mapsto x_1 \\ y_1 &\mapsto y_2 \\ y_2 &\mapsto y_1 \\ A &\rightarrow \mathbb{Z} \\ x_i &\mapsto 1 \\ y_i &\mapsto 0. \end{aligned}$$

For $i = 1$ or 2 , show that the map

$$\begin{aligned} x &\mapsto x_i \\ y &\mapsto y_i \end{aligned}$$

defines an automorphism of group schemes $\text{Spec } A \rightarrow \text{Spec } \mathbb{Z}[x, y]_{x^2+y^2}$. Evidently the two automorphisms differ by the inverse map, which is an automorphism of group schemes since these group schemes are abelian. The group scheme discussed in this exercise is the Weil restriction $\text{Res}_{\mathbb{Z}[\sqrt{-1}]/\mathbb{Z}} \mathbb{G}_m$ of \mathbb{G}_m and is often denoted \mathbb{S} .

(32) Consider the diagram of \mathbb{R} -algebras

$$\mathbb{R}[x, y] \xrightarrow[x_2, y_2]{x_1, y_1} \mathbb{R}[x_1, y_1, x_2, y_2]/\langle x_1x_2 - y_1y_2 - 1, x_1y_2 + x_2y_1 \rangle.$$

Note that Spec of the codomain ring (called A in the previous exercise) is the group scheme \mathbb{S} as in (31d), and that both maps in question give the localization

of $\mathbb{R}[x, y]$ at powers of $x^2 + y^2$ (previous exercise!), so, taking Spec we have a diagram of open embeddings, and we may glue to form a scheme

$$X = \mathbb{A}_{\mathbb{R}}^2 \coprod_{\text{Spec } A} \mathbb{A}_{\mathbb{R}}^2$$

over \mathbb{R} covered by two copies of $\mathbb{A}_{\mathbb{R}}^2$. (X is the Weil restriction of $\mathbb{P}_{\mathbb{C}}^1$ along $\mathbb{R} \rightarrow \mathbb{C}$.)

- (a) These are both consequences of being a Weil restriction, but show that there is a natural bijection $\mathbb{P}_{\mathbb{C}}^1(\mathbb{C}) \cong X(\mathbb{R})$ and a closed embedding $\mathbb{P}_{\mathbb{R}}^1 \hookrightarrow X$.
- (b) In fact, show that the bijection on points is realized by a morphism of locally ringed spaces

$$(X(\mathbb{R}), \mathcal{O}_X|_{X(\mathbb{R})} \otimes_{\mathbb{R}} \mathbb{C}) \rightarrow (\mathbb{P}_{\mathbb{C}}^1(\mathbb{C}), \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}|_{\mathbb{P}_{\mathbb{C}}^1(\mathbb{C})})$$

(in the codomain we just discard the generic point; it may be foolish to even write notation for the restriction of sheaves since $\mathbb{P}_{\mathbb{C}}^1$ and $\mathbb{P}_{\mathbb{C}}^1(\mathbb{C})$ have the same open sets!).

- (c) Show that X is a smooth surface over \mathbb{R} , but not proper over $\text{Spec } \mathbb{R}$.

- (33) Let k be a field, V a k vector space. Regard V as a scheme over k by taking the spectrum of the symmetric algebra on V^{\vee} . Then V represents the (contravariant) functor from schemes over k to sets

$$Y \mapsto \Gamma(Y, \mathcal{O}_Y) \otimes_k V$$

(naturally in V).

- (34) Give an example of a smooth Deligne-Mumford stack with singular coarse moduli space and an example of a singular DM stack with smooth coarse moduli space.
- (35) Let $\mathcal{X} \rightarrow \mathcal{C}$ be a category fibered in groupoids.
- (a) If the diagram $C \rightarrow D \leftarrow E$ in \mathcal{C} has an inverse limit in \mathcal{C} (that is, the fibered product $C \times_D E$ exists in \mathcal{C}), then any diagram lying over it in \mathcal{X} has an inverse limit.
- (b) The functor $\mathcal{X} \rightarrow \mathcal{C}$ takes cartesian squares in \mathcal{X} to cartesian squares in \mathcal{C} .
- (c) If \mathcal{C} has a topology, then \mathcal{X} inherits a topology by declaring $\{x_i \rightarrow x\}$ to be a cover in \mathcal{X} if its image in \mathcal{C} is a cover with respect to the topology on \mathcal{C} . If C is the representable CFG associated to an object of \mathcal{C} , then any morphism $C \rightarrow \mathcal{X}$ of CFGs (which we view as a functor from the slice category $\mathcal{C}/C \rightarrow \mathcal{X}$ commuting with the functors to \mathcal{C}) is a morphism of sites when \mathcal{X} has the topology induced from \mathcal{C} as above.

- (36) Show that there is no non-constant map $\mathbb{P}^2 \rightarrow \mathbb{P}^1$.

- (37) Let \mathcal{C} be a category with fibered products (and a terminal object $*$).

- (a) Show that the pullback of a categorical monomorphism is a monomorphism.
- (b) Suppose \mathcal{C} has coproducts. Show that any group G naturally gives rise to a *discrete* group object $G = \coprod_{g \in G} *$.
- (c) Let $G = (G, e : * \rightarrow G, m : G \times G \rightarrow G, I : G \rightarrow G)$ be a group object of \mathcal{C} and let $a : G \times X \rightarrow X$ be an action. For any object Z of \mathcal{C} , show that the sets

$$\{f \in \text{Hom}(Z, X) : a(\text{Id}_G \times f) = f\pi_2\}$$

and

$$\{f \in \text{Hom}(Z, X) : \forall Y \in \mathcal{C} \forall g \in \text{Hom}(Y, G) \ a(g \times f) = f\pi_2\}$$

are equal. Call this set $X^G(Z)$.

- (d) Show that the functor $Z \mapsto X^G(Z)$ is represented by a subobject X^G of X , the *fixed locus* of G . *Hint:* Given any morphism $f : X \rightarrow Y$, by the universal property of the fibered product, the pair of maps (Id_X, f) give rise to *graph morphism* $\Gamma_f : X \rightarrow X \times Y$, which must be a categorical monomorphism because $\pi_1 \Gamma_f = \text{Id}_X$.
- (e) It should be possible to similarly define the fixed locus corresponding to any morphism $g : Y \rightarrow G$, but I had trouble with this.
- (f) It should also be possible to define the stabilizer subgroup object G_f of any morphism $g : Y \rightarrow X$, but I also had trouble with this. Of course it is trivial if G is discrete: G_f is just the subobject of the coproduct given by the coproduct over the $g \in G$ such that $gf = f$.
- (38) Let $a : G \times X \rightarrow X$ be an action of a group scheme on a scheme. Let \mathcal{F} be a sheaf on X . What is a G -action on \mathcal{F} ? You should be defining a descent datum for \mathcal{F} along the morphism of groupoid fibrations $X \rightarrow [X/G]$ (the latter will be a stack under mild assumptions on G). At least do this in the case where G is discrete, so that there is essentially no difference between the group scheme G and the group $\text{Hom}(*, G)$. If there is a categorical quotient X/G (is this the same thing as a coarse moduli space for $[X/G]$ which happens to be representable by a scheme instead of an algebraic space?), then show that a sheaf \mathcal{F} on X/G naturally gives rise to a G -action on $\pi^* \mathcal{F}$.
- (39) Specialize to the case where X is finite type over a field k and $G = \text{Spec} \bigoplus_{g \in G} k$ is finite and discrete. Let \mathcal{F} be a G -sheaf on X . Let x be a geometric point of X . Notice that the stabilizer group G_x (the stabilizer of the morphism $x \rightarrow X$) acts on the vector space $x^* \mathcal{F}$. Suppose this action is trivial for every geometric point. Show that \mathcal{F} is pulled back from X/G . (This might not be right.)
- (40) Let X be the (affine) nodal cubic $y^2 = x^2(x-1)$, $f : Y \cong \mathbb{A}^1 \rightarrow X$ its normalization.
 (a) Describe the fiber product $Y \times_X Y$, and show/observe $\text{Pic}(Y \times_X Y) = 0$. Let $\pi_1, \pi_2 : Y \times_X Y \rightarrow Y$ denote the projections.
 (b) Given $\alpha \in \text{Pic}(X) = \mathbb{G}_m$, describe the identification $\pi_1^* \mathcal{O}_Y \xrightarrow{\sim} \pi_2^* \mathcal{O}_Y$. (It's a collection of elements from \mathbb{G}_m – which ones?)
- (41) Let X be a nonsingular variety over an algebraically closed field k .
 (a) Show that a degree 1 surjective étale map $f : Y \rightarrow X$ is an isomorphism.
 (b) Show that \mathbb{P}^1 has no non-trivial étale covers.
 (c) Show that \mathbb{P}^n has no non-trivial étale covers. *Hint:* Induct on n .
 (d) Assume $\text{char } k = 0$. Show that if X is rationally connected then its algebraic fundamental group is trivial.
- (42) Let X be a noetherian scheme, $j : U \hookrightarrow X$ the inclusion of an open subscheme, Z the closed complement, \mathcal{I} an ideal sheaf defining Z .
 (a) Show that U and Z are quasi-compact (this is just a general fact about noetherian topological spaces).
 (b) If Z is Cartier in X , show that for any quasi-coherent sheaf \mathcal{F} on U ,

$$R^p j_* \mathcal{F} = \begin{cases} j_* \mathcal{F}, & p = 0 \\ 0, & p > 0. \end{cases}$$

i.e. $j_* : \mathbf{Qco}(U) \rightarrow \mathbf{Qco}(X)$ is exact. *Hint:* Just show j is an affine morphism, which is a local question on X , so you can assume $X = \text{Spec } A$ is affine and $Z = \text{Spec } A/a$ for some $a \in A$. In fancy language, j is Spec_X of the localization

morphism

$$\mathcal{O}_X \rightarrow S^{-1}\mathcal{O}_X$$

of \mathcal{O}_X -algebras, with S given by the multiplicative subsheaf \mathcal{S} .

- (c) If $U = \mathbb{A}^2 \setminus \{0\}$ and j is the evident inclusion into $X = \mathbb{A}^2 = \text{Spec } \mathbb{Z}[x, y]$, then show that $R^1j_*\mathcal{O}_U \neq 0$. Note $\Gamma_U = \Gamma_Xj_*$ so $\mathbf{R}\Gamma_U = \mathbf{R}\Gamma_X\mathbf{R}j_*$, and $\mathbf{R}\Gamma_XF^\bullet = \Gamma_XF^\bullet$ if F^\bullet has quasi-coherent cohomology, so this is equivalent to saying $H^1(U, \mathcal{O}_U) \neq 0$.
- (d) Cover U by the open sets $V_0 := \{x = 0\} \subset \mathbb{A}^2$ and $V_1 := \{y = 0\} \subset \mathbb{A}^2$. Show that

$$\mathbf{R}\Gamma_U\mathcal{O}_U = \mathbf{R}\Gamma_X\mathbf{R}j_*\mathcal{O}_U$$

is given by

$$\mathbb{Z}[x, y]_x \oplus \mathbb{Z}[x, y]_y \rightarrow \mathbb{Z}[x, y]_{xy}$$

in $D^b(\mathbf{Ab})$. Show that H^0 of this complex is

$$\Gamma(U, \mathcal{O}_U) = \mathbb{Z}[x, y] \quad (= H^0(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2}))$$

and H^1 of this complex is the free abelian group on

$$\{x^m y^n : m, n < 0\}.$$

Hint: Introduce gradings.

- (e) Generalize this to $U = \mathbb{A}^n \setminus \{0\}$, $X = \mathbb{A}^n$.
- (f) Recall [EGA I §9.4] that any coherent sheaf \mathcal{F} on U can be extended to a coherent sheaf $\overline{\mathcal{F}}$ on X . For a coherent sheaf \mathcal{F} on U , define

$$j_!\mathcal{F} := \varprojlim \mathcal{I}^n \mathcal{F},$$

which is a pro-object in the category of coherent sheaves on X . Show that this is independent of the choice of lifting $\overline{\mathcal{F}}$ and the choice of ideal sheaf \mathcal{I} defining Z .

- (g) Show that the natural map

$$j_!\mathcal{F} \rightarrow j_*\mathcal{F}$$

is monic. In this and the following, you can take

$$\text{Hom}_{\text{pro } \mathcal{C}}(\varprojlim A_i, B) = \varinjlim \text{Hom}_{\mathcal{C}}(A_i, B)$$

as the definition of morphisms from a pro-object $\varprojlim A_i$ in a category \mathcal{C} to an object B of \mathcal{C} .

- (h) For $\mathcal{G} \in \mathbf{Qco}(X)$, show that there is a natural bijection

$$\text{Hom}_U(\mathcal{F}, j^*\mathcal{G}) \cong \text{Hom}_X(j_!\mathcal{F}, \mathcal{G}),$$

so $j_!$ is “left adjoint” to j^* (there are quotes because the codomain of $j_!$ isn’t the same as the domain of j^*). This $j_!$ is called “extension by zero” by Deligne and plays the coherent analogue of push-forward with proper support in the theory of Verdier duality for locally compact topological spaces.

- (i) Conclude that $j^* : \mathbf{Qco}(X) \rightarrow \mathbf{Qco}(U)$ takes injectives to injectives using general nonsense about adjoint functors and injectives.

- (43) Let $X = \text{Spec } k[x, y]/\langle y^2 = x^2(x+1) \rangle$ be the (affine) *nodal cubic curve* (see index of [Hartshorne]), $Y = \text{Spec } k[u] \cong \mathbb{A}_k^1$, $f : Y \rightarrow X$ the normalization map (Spec of the ring map $x \mapsto u^2 - 1, y \mapsto u(u^2 - 1)$).

- (a) Describe the fiber product $Y \times_X Y$, and show that $\text{Pic}(Y \times_X Y) = 0$. Show that $\text{Aut } \mathcal{O}_{Y \times_X Y} \cong \mathbb{G}_m^3$.
- (b) Let $p_1, p_2 : Y \times_X Y \rightarrow Y$ denote the projections. An $\alpha \in \text{Pic}(X)$ determines an automorphism of $\mathcal{O}_{Y \times_X Y}$; describe the resulting embedding $\mathbb{G}_m \hookrightarrow \mathbb{G}_m^3$.
- (44) **Cartan-Eilenberg resolutions.** Let B be a ring, b an element of B . Let

$$L = [B \xrightarrow{b} B]$$

be the “Koszul complex” of b , where the terms are placed in degrees $-1, 0$. Let $B \rightarrow A$ be a surjection of rings whose kernel contains b .

- (a) Let P be a projective B module and let $d : P \rightarrow \text{Ker } b$ be a surjective B module morphism. Regard d as a B module map $d : P \rightarrow A$ by composing with the inclusion $\text{Ker } b \hookrightarrow B$ and the map $B \rightarrow A$. Show that the ideal

$$I = \{a \in A : ad = 0\}$$

is independent of the choice of such (P, d) and that, regarding I as a B module by restriction of scalars along $B \rightarrow A$, we have a natural isomorphism $I = \text{Ext}_B^1(H^0(L), A)$.

- (b) Show that $\mathbb{E}\text{xt}_B^1(L, A) \cong A$, and that the natural monomorphism

$$\text{Ext}_B^1(H^0(L), A) \hookrightarrow \mathbb{E}\text{xt}_B^1(L, A)$$

is identified naturally with the inclusion $I \subseteq A$.

Hint: Let $P_\bullet \rightarrow H^{-1}(L)$ be a projective resolution. It extends to give a projective resolution

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \xrightarrow{b} B \longrightarrow H^0(L) \longrightarrow 0$$

of $H^0(L)$. Observe that

$$\begin{array}{ccccc}
 \vdots & & & & \vdots \\
 \begin{pmatrix} d_3 & -1 \\ 0 & d_2 \end{pmatrix} \downarrow & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} d_2 & -1 \\ 0 & d_1 \end{pmatrix} \\
 P_2 \oplus P_1 & \xrightarrow{\quad} & P_1 \oplus P_0 & & \\
 \begin{pmatrix} d_2 & 1 \\ 0 & d_1 \end{pmatrix} \downarrow & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} d_1 & 1 \\ 0 & d_0 \end{pmatrix} \\
 P_1 \oplus P_0 & \xrightarrow{\quad} & P^0 \oplus B & & \\
 \begin{pmatrix} d_1 & -1 \\ 0 & d_0 \end{pmatrix} \downarrow & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} d_0 & -1 \\ 0 & b \end{pmatrix} \\
 P_0 \oplus B & \xrightarrow{\quad} & B \oplus B & & \\
 \begin{pmatrix} d_0 & 1 \end{pmatrix} \downarrow & & & & \downarrow \begin{pmatrix} b & 1 \end{pmatrix} \\
 B & \xrightarrow{\quad b \quad} & B & &
 \end{array}$$

corresponds to the inclusion $I \hookrightarrow H^1(\text{Tot})$ induced by including $I = 0 \oplus I$ in the $(0, 1)$ term of the above double complex (note this is in $\text{Ker } d_v^{0,1} \cap \text{Ker } d_h^{0,1}$ and the map clearly induces a monomorphism on $H^1(\text{Tot})$ because nothing of this form is in $\text{Im } d_T^0$).

The E_2 term of the spectral sequence where we first take horizontal cohomology is

$$E_2^{p,q} = \text{Ext}^p(H^{-q}(L), A),$$

which clearly vanishes except when $(p, q) \in [0, \infty) \times [0, 1]$. The only interesting d_2 differential is therefore:

$$d_2^{p,1} : \text{Ext}^p(H^{-1}(L), A) \rightarrow \text{Ext}^{p+2}(H^0(L), A).$$

(c) Show that this map is surjective when $p = 0$ and an isomorphism for $p > 0$.

Hint: Identify this map with the map on cohomology induced by the map of complexes

$$\begin{array}{ccccccc} A & \xrightarrow{0} & A & \xrightarrow{d_0^*} & P^0 & \xrightarrow{d_1^*} & P^1 & \xrightarrow{d_2^*} & \dots \\ & & & & \uparrow \text{Id} & & \uparrow \text{Id} & & \\ & & & & P^0 & \xrightarrow{d_1^*} & P^1 & \xrightarrow{d_2^*} & \dots \end{array}$$

obtained by applying $\text{Hom}(_, A)$ to the projection

$$\begin{array}{ccccccc} \dots & \longrightarrow & P_1 & \xrightarrow{d_1} & P^0 & & \\ & & \uparrow \text{Id} & & \uparrow \text{Id} & & \\ \dots & \longrightarrow & P_1 & \xrightarrow{d_1} & P^0 & \xrightarrow{d_0} & B & \xrightarrow{b} & B. \end{array}$$

(45) **Cotangent complex I.** Let A be a ring and let $B = A[x]/x^k$. As usual, we view A as a B module throughout via the ring map $B \rightarrow A$ sending x to 0 (i.e. we identify A with B/xB). Note that $B \rightarrow A$ has a natural section $s : A \rightarrow B$ as a map of A algebras, but this section is not a map of B modules.

(a) Show that the cotangent complex $\mathbb{L}_{B/A}$ is isomorphic in $D(B)$ to the complex

$$L = [B \xrightarrow{kx^{k-1}} B],$$

where the terms are placed in degrees $-1, 0$. (The complex L might be called the *Koszul complex* of $kx^{k-1} \in B$.) *Hint:* Since B is given by the regular ideal (x^k) in the free A algebra $A[x]$ ($\text{Spec } B$ is a Cartier divisor in \mathbb{A}_A^1), the cotangent complex $\mathbb{L}_{B/A}$ is of perfect amplitude $[-1, 0]$ and is hence in particular isomorphic to its truncation

$$\mathbb{L}_{B/A} = \tau_{\geq -1} \mathbb{L}_{B/A} = [(x^k)/(x^{2k}) \xrightarrow{d} \Omega_{A[x]/A} \otimes_{A[x]} B].$$

(b) Show that $\mathbb{E}xt_B^1(\mathbb{L}_{B/A}, A) = A$. *Hint:* See (44b).

(c) Label the boundary maps below with elements of B so that

$$\dots \longrightarrow B \longrightarrow B \longrightarrow B \longrightarrow B \longrightarrow A$$

is a free resolution of A .

(d) Let M be an A module and let

$$\cdots \longrightarrow A^{r_2} \xrightarrow{d^2} A^{r_1} \xrightarrow{d^1} A^{r_0} \longrightarrow M$$

be a free resolution of M (the direct sums A^{r_i} might be infinite). Use the previous part to promote this to a free resolution of M , now regarded as a B module via restriction of scalars along $B \rightarrow A$. *Hint:* Use double complexes to avoid concerning yourself with sign conventions. Lift the coefficients of the matrices d^i to B via s .

(e) Let $J := \text{Ann}(k) \subset A$ be the kernel of multiplication by k in A . Show that $\text{Ext}_B^1(\Omega_{B/A}, A) = \text{Ann } J$ and that the natural inclusion

$$\text{Ext}_B^1(\Omega_{B/A}, A) \hookrightarrow \mathbb{E}\text{xt}_B^1(\mathbb{L}_{B/A}, A)$$

is identified with the inclusion of $\text{Ann } J$ in A (of course, $\text{Ann } J$ is viewed as a B module by restriction of scalars throughout). *Hint:* Identify $\text{Ann } J$ with the ideal $I \subset A$ of (44b). Or, what amounts to the same thing, let $\bar{j} = \{j_s : s \in S\}$ be a set of generators for J and compute $\text{Ext}_B^1(\Omega_{B/A}, A)$ using a projective resolution of the form

$$\cdots \longrightarrow B \oplus (\oplus_S B) \xrightarrow{(x, \bar{j})} B \xrightarrow{kx^{k-1}} B.$$

(46) **Cotangent complex II.** This is a simple example of a ring map which is not l.c.i. but for which the entire cotangent complex can easily be computed. Let $A := \mathbb{Z}[x]/x^{n+1}$ be the coordinate ring of the n^{th} infinitesimal neighborhood of the origin. Assume $n > 0$. Show that the cotangent complex $\mathbb{L}_{\mathbb{Z}/A} \in D(\mathbf{Ab})$ of the map $A \rightarrow \mathbb{Z}$ given by $x \mapsto 0$ is quasi-isomorphic to the complex $[\mathbb{Z} \rightarrow \mathbb{Z}]$ with zero differential of perfect amplitude $[-2, -1]$. i.e. $\mathbb{L}_{\mathbb{Z}/A} = \mathbb{Z}[1] \oplus \mathbb{Z}[2]$. *Hint:* First show that it suffices to prove that $h^{-1}(\mathbb{L}) \cong h^{-2}(\mathbb{L}) \cong \mathbb{Z}$ and that all other cohomology groups of $\mathbb{L}_{\mathbb{Z}/A}$ vanish (going from the statement about the cohomology of the complex to the quasi-isomorphism type of the complex is easy if the cohomology is projective). To prove the latter, consider the distinguished triangle of cotangent complexes associated to the ring maps $\mathbb{Z}[x] \rightarrow \mathbb{Z}[x]/x^{n+1} \rightarrow \mathbb{Z}$. *Remark:* There is a theorem (Quillen, Avramov) to the effect that a (noetherian, say) ring map is l.c.i. iff it has finite flat dimension and its cotangent complex has finite flat dimension iff its cotangent complex is of perfect amplitude $\subseteq [-1, 0]$. The situation here is: $\mathbb{L}_{\mathbb{Z}/A}$ has finite flat dimension (clearly), the ring map is not l.c.i. (x is not regular in $\mathbb{Z}[x]/x^n$), and \mathbb{Z} has infinite flat dimension as a $\mathbb{Z}[x]/x^n$ -module.

(47) **Cotangent complex III.** Let A be a ring and let $B = A[x]/x^k$. In this exercise, we will construct an isomorphism of B modules

$$a : \text{Exal}_A(B, A) \xrightarrow{\cong} A$$

from first principles. Suppose

$$\begin{array}{ccccccc} \underline{C} & = & 0 & \longrightarrow & A & \xrightarrow{i} & C & \xrightarrow{f} & B & \longrightarrow & 0 \\ & & & & & & & & \uparrow & & \\ & & & & & & & & A & & \end{array}$$

is a (square zero) algebra extension. Set $y = i(1) \in C$. Show that:

(a) $y^2 = 0$ in C and $y \cdot f^{-1}(x) = 0$ in C , meaning $y\bar{x} = 0$ for every $\bar{x} \in f^{-1}(x)$.

- (b) There is an $a = a(C) \in A$ such that $\bar{x}^k = ay$ in C for every $\bar{x} \in f^{-1}(x)$.
(c) For any $\bar{x} \in f^{-1}(x)$, the A module morphism $(1, \bar{x}, \bar{x}^2, \dots, \bar{x}^{k-1}, y) : A^{k+1} \rightarrow C$ is an isomorphism.
(d) Conclude from this that, for any $\bar{x} \in f^{-1}(x)$,

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{y} & A[x, y]/(xy, y^2, x^k - ay) & \longrightarrow & A[x]/x^2 \longrightarrow 0 \\
& & \parallel & & \downarrow x \mapsto \bar{x} & & \parallel \\
0 & \longrightarrow & A & \xrightarrow{i} & C & \xrightarrow{f} & B \longrightarrow 0
\end{array}$$

is a morphism in $\underline{\text{Exal}}_A(B, A)$ (hence an isomorphism). (Different choices of $\bar{x} \in f^{-1}(x)$ will yield isomorphisms differing by an automorphism of the top line in $\underline{\text{Exal}}_A(B, A)$.) You should first prove that the top line actually is an element of $\text{Exal}_A(B, A)$ for any $a \in A$. Conclude that $a : \text{Exal}_A(B, A) \rightarrow A$ is an isomorphism.

- (e) If $a \in A^*$ is a unit, then show that the top line above is isomorphic in $\underline{\text{Exal}}_A(B, A)$ to

$$\begin{array}{ccccccc}
0 & \longrightarrow & A & \xrightarrow{a^{-1}x^k} & A[x]/x^{k+1} & \longrightarrow & A[x]/x^k \longrightarrow 0 \\
& & & & \swarrow & & \uparrow \\
& & & & & & A
\end{array}$$

- (f) Prove, however, that $\mathbb{Z}[x]/x^3$ is not isomorphic (even just as rings) to

$$\mathbb{Z}[x, y]/(xy, y^2, x^2 - 2y).$$

- (g) Assume that k is regular in A , so that $\text{Ann}(k) = 0$ and we have

$$\begin{aligned}
\text{Ext}_B^1(\Omega_{B/A}, A) &= \text{Ann Ann}(k) \\
&= A \\
&= \mathbb{E}xt_B^1(\mathbb{L}_{B/A}, A)
\end{aligned}$$

(c.f. (45)). Prove that the isomorphism a constructed above coincides with the isomorphism

$$\mathbb{E}xt_B^1(\mathbb{L}_{B/A}, A) \cong \text{Exal}_A(B, A)$$

provided by the Fundamental Theorem of the Cotangent Complex (III.1.2.1).

Hint: The restriction

$$\text{alg} : \text{Ext}_B^1(\Omega_{B/A}, A) \hookrightarrow \text{Exal}_A(B, A)$$

of the Fundamental Theorem isomorphism to $\text{Ext}_B^1(\Omega_{B/A}, A)$ can be described rather elementarily as follows (see (III.1.1.8.2) in Illusie's book *Complexe Cotangent et Déformations*). Given an extension

$$\underline{E} = (0 \longrightarrow A \xrightarrow{i} E \xrightarrow{f} \Omega_{B/A} \longrightarrow 0),$$

the algebra extension $\underline{C} = \text{alg}(\underline{E})$ is defined (up to unique isomorphism) by requiring the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{(0,i)} & B \oplus E & \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}} & B \oplus \Omega_{B/A} \longrightarrow 0 \\
 & & \parallel & & \uparrow & & \uparrow (1, d_{B/A}) \\
 0 & \longrightarrow & A & \longrightarrow & C & \longrightarrow & B \longrightarrow 0
 \end{array}$$

to commute (i.e. by taking the pullback of the extension in the top row). Note that C inherits its A algebra structure from the A algebra structure of trivial square zero extension (of B by E) on $B[E]$.

- (h) Check that the multiplication in $B[E]$ preserves C and that the B module map $C \rightarrow B$ appearing in the above diagram is actually a map of A algebras.

You should now explicitly write down an extension \underline{E} of $\Omega_{B/A}$ by A corresponding to $1 \in A$ under $\text{Ext}_B^1(\Omega_{B/A}, A) \cong A$ (it should be sort of obvious how to do this just by looking at the complex L , but you should be clear about where you need regularity of k in A) and check that $a(\text{alg}(\underline{E})) = 1$.

- (48) **Cotangent complex IV.** Let $f : A \rightarrow B$ be a surjective map of rings (or ring objects of a topos if you like) with kernel I . Directly construct an isomorphism

$$\text{Exal}_A(B, M) \cong \text{Hom}_B(I/I^2, M),$$

natural in the B module M . Note I/I^2 is regarded as a B module as usual by setting $b \cdot i = \bar{b}i$ where $\bar{b} \in f^{-1}(b)$ (the choice is irrelevant mod I^2). Show that $\text{Id} \in \text{Hom}_B(I/I^2, I/I^2)$ corresponds to the first infinitesimal neighborhood:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & I/I^2 & \longrightarrow & A/I^2 & \longrightarrow & B \longrightarrow 0 \\
 & & & & \swarrow & & \uparrow \\
 & & & & & & A
 \end{array}$$

Solution:

Constructing the map

$$\text{Exal}_A(B, M) \rightarrow \text{Hom}_B(I/I^2, M)$$

is easy: Given an algebra extension

$$\underline{C} = 0 \longrightarrow M \longrightarrow C \longrightarrow B \longrightarrow 0$$

\swarrow
 $\uparrow f$
 A

and an $i \in I = \text{Ker } f$, commutativity implies $j(i) \in \text{Ker } g = M$. For $i_1, i_2 \in I$, $j(i_1 i_2) = j(i_1)j(i_2) = 0$ since both $j(i_1)$ and $j(i_2)$ are in M and the latter is square zero in C . Thus we get a map $I/I^2 \rightarrow M$ which is easily seen to be B

linear (everything is inherited from C). It is obvious that this map takes the first infinitesimal neighborhood to the identity map $\text{Id} \in \text{Hom}_B(I/I^2, I/I^2)$.

To construct an inverse

$$\text{Hom}_B(I/I^2, M) \rightarrow \text{Exal}_A(B, M),$$

pushout the first infinitesimal neighborhood exact sequence (note $B = A/I$)

$$0 \longrightarrow I/I^2 \longrightarrow A/I^2 \longrightarrow B \longrightarrow 0$$

along $I/I^2 \rightarrow M$. It is also obvious that this takes $\text{Id} \in \text{Hom}_B(I/I^2, I/I^2)$ to the first infinitesimal neighborhood extension!

- (49) **Cotangent complex V.** Let k be a ring (or sheaf of rings) and let $A \rightarrow B$ be a surjection of k -algebras with kernel I . Consider the image of the first infinitesimal neighborhood extension (see the previous problem) under the “forgetful” map:

$$\text{Exal}_B(A, I/I^2) \rightarrow \text{Exal}_k(A, I/I^2).$$

By the naturality of the Fundamental Theorem of the Cotangent Complex, this “forgetful” map is identified with the natural map

$$\text{Ext}_B^1(\mathbb{L}_{B/A}, I/I^2) \rightarrow \text{Ext}_B^1(\mathbb{L}_{B/k}, I/I^2).$$

Hence, viewed as a derived category morphism, the first infinitesimal neighborhood corresponds to the composition $\mathbb{L}_{B/k} \rightarrow \mathbb{L}_{B/A} \rightarrow \tau_{\geq -1}\mathbb{L}_{B/A} = I/I^2[1]$.

Let $A = \mathbb{Z}[x, y]$ and let $B = \mathbb{Z}[x, y]/xy$, so $I = (xy)$ is the kernel of the natural surjection $A \rightarrow B$ and $B_1 := \mathbb{Z}[x, y]/x^2y^2$ is the first infinitesimal neighborhood of $N := \text{Spec } B$ in $\mathbb{A}^2 = \text{Spec } A$. Since N is a Cartier divisor in the smooth scheme \mathbb{A}^2 , $\mathbb{L}_B = [I/I^2 \rightarrow \Omega_{\mathbb{A}^2}|_{\text{Spec } B}]$. Observe that the extension $B_1 \rightarrow B$ can also be written as

$$0 \longrightarrow B \xrightarrow{z} \mathbb{Z}[x, y, z]/(xy - z, z^2) \xrightarrow{z \mapsto 0} B \longrightarrow 0,$$

which is the deformation that “smooths the node”.

- (a) Show that

$$\begin{aligned} \mathbb{L}_B &= \left[\begin{pmatrix} x \\ y \end{pmatrix} : B \rightarrow B^2 \right] \\ &= \Omega_B. \end{aligned}$$

(Note $I/I^2 \cong B$ is free on the generator $xy \in I/I^2$.)

- (b) Use the resolution $B \rightarrow B^2$ of Ω_B to show that $\text{Ext}^1(\Omega_B, B) = \mathbb{Z}$ (regarded as a B -module via the ring map $B \rightarrow \mathbb{Z}$ killing x and y).
- (c) Explicitly describe the extension in $\text{Ext}_B^1(\Omega_B, B)$ corresponding to the first infinitesimal neighborhood of N in \mathbb{A}^2 and show that it generates $\text{Ext}^1(\Omega_B, B) \cong \mathbb{Z}$. In particular, observe that this extension is nontrivial. In fact, $\mathbb{Z}[x, y]/x^2y^2$ is not even isomorphism as a ring to the trivial extension $\mathbb{Z}[x, y, z]/(xy, z^2)$.¹
Hint: See the previous problem. Note that $\mathbb{L}_{B/k} \rightarrow \tau_{\geq 1}I/I^2[1]$ is just the obvious projection map $[I/I^2 \rightarrow \Omega_A \otimes_A B] \rightarrow I/I^2[1]$.

¹However, the associated graded of the (xy) -adic filtration on $\mathbb{Z}[x, y]/(x^2y^2)$ is $\mathbb{Z}[x, y, z]/(xy, z^2)$, and we will see in Problem 59 that this in particular gives a flat family over \mathbb{A}^1 with general fiber $\mathbb{Z}[x, y]/(x^2y^2)$ and special fiber $\mathbb{Z}[x, y, z]/(xy, z^2)$.

(d) In general, $B_n := \mathbb{Z}[x, y]/x^{n+1}y^{n+1}$ is the (coordinate ring of the) n^{th} infinitesimal neighborhood of N in \mathbb{A}^2 . B_n is a square zero ring extension of B_{n-1} by $I^n/I^{n+1} \cong B\langle x^n y^n \rangle$ and is hence classified by an element of $\text{Ext}_{B_{n-1}}^1(\mathbb{L}_{B_n}, B)$. Describe \mathbb{L}_{B_n} and each of these extensions.

- (50) **Cotangent complex VI.** Let $f : X \rightarrow Y$ be a flat morphism of schemes. Let $Y \hookrightarrow Y[\epsilon]$ denote Spec_Y of the map $\mathcal{O}_Y[\epsilon]/\epsilon^2 \rightarrow \mathcal{O}_Y$ sending ϵ to 0. Show that there is a natural bijection between isomorphism classes of cartesian diagrams of schemes

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ f \downarrow & & \downarrow f' \\ Y & \hookrightarrow & Y[\epsilon] \end{array}$$

with f' flat and \mathcal{O}_Y algebra extensions of \mathcal{O}_X by \mathcal{O}_X . (One issue is to show that in any square zero ring extension $0 \rightarrow M \rightarrow \mathcal{O}_{X'} \rightarrow \mathcal{O}_X \rightarrow 0$ with M a quasi-coherent \mathcal{O}_X -module, the ringed space $(X, \mathcal{O}_{X'})$ is actually a scheme. Reduce to the case where X is affine and use the vanishing of higher cohomology of a quasi-coherent sheaf on an affine scheme.) Such diagrams are typically called *deformations* of f and are classified by $\text{Ext}^1(\mathbb{L}_{X/Y}, \mathcal{O}_X)$ according to the fundamental theorem of the cotangent complex.

- (51) Recall that an element of a ring is called *regular* if it is not a zero divisor. A sequence of elements x_1, \dots, x_n of a ring A is called a *regular sequence* if, for each i , x_i is a regular nonunit in $A/\langle x_1, \dots, x_{i-1} \rangle$. The maximum length of a regular sequence in A is called the *depth* of A (notation: $d(A)$). It may be infinite. A ring A of finite Krull dimension (notation: \dim) is called *Cohen-Macaulay* if $d(A) = \dim(A)$. For example, if $\dim(A) = 1$, then it fails to be CM iff any element of A is either a unit or a zero divisor.
- Show that $d(A) \leq \dim(A)$. *Hint:* Hauptidealsatz.
 - Show that a local complete intersection is CM.
 - Show that a local ring A is regular iff its residue field is a complete intersection in its spectrum.
 - Show that $\mathbb{Z}[x_1, \dots, x_n]/\langle \{x_i x_j : 1 \leq i < j \leq n\} \rangle$ (the union of the axes in \mathbb{A}^n) is a Cohen-Macaulay curve, but not a local complete intersection unless $n \leq 2$.
 - Show that the union of the axes in \mathbb{A}^n is the direct limit of the diagram

$$\begin{array}{ccc} \text{Spec } \mathbb{Z} & \longrightarrow & \text{Spec } \mathbb{Z}[x_1] \\ & \searrow & \downarrow \\ & & \text{Spec } \mathbb{Z}[x_2] \\ & & \dots \\ & \searrow & \downarrow \\ & & \text{Spec } \mathbb{Z}[x_n] \end{array}$$

(each map is $x_i \mapsto 0$) in the category of schemes. (c.f. the series of exercises on gluing schemes along closed subschemes). This can also be expressed using

inverse limits in the category of rings as

$$\text{Spec} \{(f_1, \dots, f_n) \in \mathbb{Z}[x_1] \times \dots \times \mathbb{Z}[x_n] : f_1(0) = \dots = f_n(0)\}.$$

- (f) Show that $A = \mathbb{Z}[x, y]/\langle y^2, xy \rangle$ is not Cohen-Macaulay at the origin (an embedded point).
- (g) Regard A as a module over $\mathbb{Z}[x, y]$ via the projection, so that the support of A is the non-CM curve from the previous part. Show that the $\mathbb{Z}[x, y]$ -submodule of A generated by y is nonzero, but has zero dimensional support (i.e. A is not a *pure* $\mathbb{Z}[x, y]$ -module). (The purity condition arises in the stable pairs curve counting theory of Pandharipande-Thomas.)
- (h) Give an example of a curve which fails to be CM at a reduced point, or prove this doesn't happen.
- (52) Let $X = \text{Spec} \mathbb{Z}[x, y]/\langle xy(x - y) \rangle$ be three lines in the plane \mathbb{A}^2 intersecting at the origin. X is a Cartier divisor in a smooth scheme, so certainly it is Cohen-Macaulay. Let Y be the three axes in \mathbb{A}^3 .
- (a) Observe that the inclusions of each of the three lines in X define three maps $\mathbb{A}^1 \rightarrow X$ which agree at the origin of each \mathbb{A}^1 , so by (51e) and the universal property of direct limits, they define a map $Y \rightarrow X$. Show that this is the semi-normalization of X . Observe that the seminormalization of X is not LCI (51d), but X certainly is.
- (b) Show that the inclusion of the three lines $U := \mathbb{A}^1 \amalg \mathbb{A}^1 \amalg \mathbb{A}^1$ into the three axes in \mathbb{A}^3 defines a cover $U \rightarrow Y$ in the canonical topology (which is not flat and has no section, even flat locally near the origin!), but the induced map $U \rightarrow X$ (the inclusion of the three lines in X) is not a canonical cover. *Hint:* Show that the natural map $U \times_Y U \rightarrow U \times_X U$ is an isomorphism (each component of this depends on the inclusion of at most two lines, where everything is well-behaved), so these can't both be canonical covers, else giving a map out of X would be the same as giving a map out of Y and we would conclude $X \cong Y$, which is false!
- (53) Consider the ring

$$A := \mathbb{Z}[x_1, \dots, x_k]/\langle \{x_i x_j : i, j \in \{1, \dots, k\}\} \rangle$$

(the coordinate ring of the first infinitesimal neighborhood of the origin in \mathbb{A}^k .) View \mathbb{Z} as an A module by identifying it with the quotient of A by the ideal generated by the x_i (the “structure sheaf of the origin”).

- (a) Write down a resolution of $\mathbb{Z} \in \mathbf{Mod}_A$ by finitely generated free A modules such that each boundary map $A^m \rightarrow A^n$ is given by a matrix each of whose entries is one of the x_i (these complexes will be infinite when $k > 1$).
- (b) Compute $\text{Ext}^i(\mathbb{Z}, A)$ for every i . *Hint:* You will find that $\text{Ext}^i(\mathbb{Z}, A) \cong \mathbb{Z}^{f(i, k)}$ for some appropriate $f(i, k) \in \mathbb{N}$.
- (c) Show that $\mathbf{R} \text{Hom}(\mathbb{Z}, A)$ is cohomologically formal (quasi-isomorphic to a complex whose boundary maps are zero).
- (d) When $k > 1$ conclude that \mathbb{Z} (viewed as an object of the derived category of A modules) is not quasi-isomorphic to its double derived dual.
- (54) **Quot schemes I.** Let k be an algebraically closed field, C a smooth proper curve over k , E a vector bundle on C . Let $\text{Quot } E$ denote the scheme parameterizing (flat families of) short exact sequences

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

of sheaves on C and let $\text{Quot}^{r,d} E$ denote the component of $\text{Quot} E$ where S has rank r and degree d . $\text{Quot}^{r,d} E$ is projective. The subsheaf S is locally free since it is a subsheaf of a locally free (coherent) sheaf and \mathcal{O}_C is a sheaf of PIDs.

- (a) Suppose $C = \mathbb{P}^1$ and $E = \mathcal{O}(a)^m \oplus \mathcal{O}(a+1)^n$ is a balanced vector bundle on \mathbb{P}^1 . Show that $\text{Quot} E$ is smooth. *Hint:* It helps if you know about the deformation/obstruction theory of quotients and you use the formal smoothness criterion. You may freely use the following fact: Given a scheme Y_0 , a SES

$$0 \rightarrow S \rightarrow \pi_2^* E \rightarrow Q \rightarrow 0$$

on $Y_0 \times C$ with Q flat over Y_0 and a square zero closed embedding $Y_0 \hookrightarrow Y$ with ideal I , there is an obstruction

$$\text{ob} \in \text{Ext}_{Y_0 \times C}^1(S, Q \otimes \pi_1^* I)$$

whose vanishing is necessary and sufficient for the existence of a sequence

$$0 \rightarrow \bar{S} \rightarrow \pi_2^* E \rightarrow \bar{Q} \rightarrow 0$$

on $Y \times C$ with \bar{Q} flat over Y restricting to the previous SES on $Y_0 \times C$.

- (b) Set $Q_{d,n} := \text{Quot}^{n,-d} \mathcal{O}_C^n$ and $Q_n := \coprod_d Q_{d,n}$. Show that the projective scheme $Q_{d,n}$ is smooth.
(c) On Q_n , the determinant of the inclusion $S \rightarrow \mathcal{O}_C^n$ gives an inclusion $\det S \rightarrow \mathcal{O}_C$. The quotient of this inclusion determines a morphism

$$\det_n : Q_n \rightarrow \text{Quot} \mathcal{O}_C = \coprod_d \text{Sym}^d C$$

restricting to a morphism $\det_{d,n} : Q_{d,n} \rightarrow \text{Sym}^d C$. Let P_1, P_2, \dots, P_l be distinct points of C , and let m_1, \dots, m_l be positive integers, so $\sum_i m_i P_i$ is a typical point of $\text{Sym} C$. Show that

$$\det^{-1}\left(\sum_i m_i P_i\right) \cong \prod_i \det^{-1}(m_i P_i)$$

(as schemes). *Hint:* Glue subsheaves over the open sets $U_i := C \setminus (\cup_{j \neq i} P_j)$.

- (d) Show that $\det_n^{-1}(dP)$ does not depend on P or C , only on d and n , hence we may denote it $X_{d,n}$. In fact, show that $X_{d,n}$ is nothing but the quot scheme parameterizing quotients

$$\oplus_n k[t]/t^M \rightarrow Q$$

of $k[t]/t^M$ modules with $\dim_k Q = d$ (this is clearly independent of the choice of $M \geq d$) discussed in (55). *Hint:* Let $N_M(P) := \text{Spec}_C \mathcal{O}_C / \mathfrak{m}_P^M \cong \text{Spec} k[t]/t^M$ be the $(M-1)^{\text{st}}$ infinitesimal neighborhood of P in C . If

$$0 \rightarrow S \rightarrow \mathcal{O}_C^n \rightarrow Q \rightarrow 0$$

is a SES on C where S has rank n (so Q is torsion) and $\dim_k H^0(C, Q) = d$ and Q is supported topologically on P , then on dimension grounds the quotient map factors (uniquely) through

$$\mathcal{O}_C^n \rightarrow \mathcal{O}_{N_M(P)}^n$$

as long as $M \geq d$ (Q is pushed forward from $N_M(P)$).

- (e) Working over $k = \mathbb{C}$, show that the Betti numbers of the smooth projective variety $Q_{2,2}$ (over the curve $C = \mathbb{P}^1$) are: $b_0 = 1, b_2 = 2, b_4 = 4, b_6 = 2, b_8 = 1$. (The cohomology ring $H^*(Q_{2,2}, \mathbb{Q})$ is computed in Section 6 of: T. Braden, L. Chen, and F. Sottile, *The equivariant Chow rings of Quot schemes*.) Try to understand the class of $X_{2,2}$ in the Grothendieck ring of varieties in terms of that of $Q_{2,2}$ by studying the stratification

$$Q_{2,2} = \det_{2,2}^{-1}(\mathrm{Sym}^2 \mathbb{P}^1 \setminus \Delta) \coprod \coprod \det_{2,2}^{-1}(\Delta).$$

You should be able to at least compute the topological Euler characteristic of $X_{2,2}$ if not its entire mixed Hodge polynomial.

- (55) **Quot schemes II: Affine Grassmannian.** Let k be an algebraically closed field and let $X_{d,n}$ denote the Quot scheme parameterizing (flat families of) short exact sequences

$$0 \rightarrow S \rightarrow (k[x]/x^M)^n \rightarrow Q \rightarrow 0$$

of $k[x]/x^M$ modules with $\dim_k Q = d$ (for any sufficiently large M ; see below).

- (a) Observe that, as suggested by the notation, this Quot scheme is independent of the choice of M as long as $M \geq d$.
- (b) Clearly $X_{d,n}$ embeds in the Grassmannian $\mathrm{Gr}(dn - d, dn)$ of d dimensional quotients of k^{dn} by forgetting the $k[x]/x^d$ module structure. Express $X_{d,n} \hookrightarrow \mathrm{Gr}(dn - d, dn)$ as a degeneracy locus of a map of vector bundles on the Grassmannian.
- (c) Let $\lambda = (\lambda_i)$ be a partition of d . Show that the locus Gr_n^λ in $X_{d,n}$ where the quotient Q admits a direct sum decomposition $Q \cong \sum_i Q_i$ (as $k[x]/x^d$ modules) with Q_i indecomposable and $\dim_k Q_i = \lambda_i$ is locally closed in $X_{d,n}$. Show that the closure of Gr_n^λ in $X_{d,n}$ is the union of the Gr_n^μ over partitions $\mu \leq \lambda$ (here \leq is the “usual” ordering of partitions, or perhaps the “usual” ordering of the transposed partitions; the partition $1 + 1 + \dots + 1$ should be the minimum element in the \leq ordering). My understanding is that Gr_n^λ is called the “affine Grassmannian”.
- (d) Show that $X_{1,n} = \mathbb{P}^{n-1}$.
- (e) Show that $\mathrm{Gr}_n^{1+1+\dots+1} \cong \mathrm{Spec} k$ is a single point corresponding to the submodule S^* of $(k[x]/x^d)^n$ generated by

$$(x, 0, \dots, 0), (0, x, 0, \dots, 0), \dots, (0, \dots, 0, x),$$

whose quotient is $Q^* = k^n$.

- (f) Compute the dimension of the tangent space to $X_{d,n}$ at the point

$$P = (0 \rightarrow S^* \rightarrow (k[x]/x^d)^n \rightarrow Q^* \rightarrow 0)$$

by using the isomorphism from deformation theory:

$$T_P X_{d,n} = \mathrm{Hom}_{k[x]/x^d}(S^*, Q^*).$$

- (g) From (55c) and (55e), we have

$$X_{2,2} = \mathrm{Gr}_2^2 \coprod \coprod \mathrm{Gr}_2^{1+1} = \mathrm{Gr}_2^2 \coprod \coprod \mathrm{Spec} k.$$

Show that Gr_2^2 is isomorphic to the total space of $\mathcal{O}_{\mathbb{P}^1}(2)$. *Hint:* Show that, for any sequence

$$0 \rightarrow S \rightarrow (k[x]/x^2)^2 \rightarrow Q \rightarrow 0$$

in Gr_2^2 , S can be generated as a $k[x]/x^2$ module by a single element (this is true of Q by definition of Gr_2^2). If $(a + bx, c + dx)$ is a generator of S , then show that the condition $\dim_k S = 2$ implies that at least one of a, c is in k^* . If, say, $a \in k^*$, then $a + bx \in (k[x]/x^2)^*$, so S can be generated by an element of the form $(1, c + dx)$. Show that this defines an open embedding $\mathbb{A}_{c,d}^2 \hookrightarrow \text{Gr}_2^2$. Similarly, the locus where S can be generated by an element of the form $(a + bx, 1)$ defines an open embedding $\mathbb{A}_{a,b}^2 \hookrightarrow \text{Gr}_2^2$. Show that these overlap in a $\mathbb{G}_m \times \mathbb{A}^1$, write down the transition function for the gluing, and recognize it as the transition function for

$$\mathcal{O}_{\mathbb{P}^1}(2) = \mathbb{A}^2 \coprod_{\mathbb{G}_m \times \mathbb{A}^1} \mathbb{A}^2.$$

- (h) Conclude that $X_{2,2}$ is a projective surface whose singular locus consists of a single point $P = \text{Gr}_2^{1+1}$ with

$$\dim_k T_P X_{2,2} = \dim_k \text{Hom}_{k[x]/x^2}(k^2, k^2) = 4.$$

- (i) Consider the embedding $X_{2,2} \hookrightarrow \text{Gr}(2, 4)$. View $X_{2,2}$ as $\text{Quot}^2(k[x]/x^2)^2$ and use $(1, 0), (x, 0), (0, 1), (0, x)$ as an ordered k basis for $k[x]/x^2$ and hence to identify $(k[x]/x^2)^2$ with k^4 as a k vector space. For $1 \leq i < j \leq 4$, let $U_{ij} \cong \mathbb{A}^4 \subset \text{Gr}(2, 4)$ be the usual chart for the Grassmannian centered at $\langle e_i, e_j \rangle$. The interesting chart (for our purposes) is $U_{2,4} \cong \mathbb{A}_{t_1, t_2, t_3, t_4}^4$ with (t_1, t_2, t_3, t_4) corresponding to

$$\langle (t_1, 1, t_2, 0), (t_3, 0, t_4, 1) \rangle \in \text{Gr}(2, 4).$$

Give generators for the ideal $I \in k[t_1, t_2, t_3, t_4]$ corresponding to the embedding

$$X_{2,2} \cap U_{2,4} \hookrightarrow U_{2,4} \cong \mathbb{A}^4.$$

Notice that the singular point $P \in X_{2,2}$ corresponds to the origin in \mathbb{A}^4 , so show that your ideal I defines a surface in \mathbb{A}^4 singular only at the origin with a 4 dimensional tangent there.

- (56) **Quot schemes III.** Continue with the notation from (54), though we will always take $C = \mathbb{P}^1$ here. The purpose of this exercise is to study a simple example where $\text{Quot } E$ is singular for a vector bundle E on \mathbb{P}^1 .

- (a) Working over $C = \mathbb{P}^1$, show that $\text{Quot } E$ is smooth whenever E has rank at most two. *Hint:* The only case not covered by parts of the previous exercise is the component $\text{Quot}^{1,d} E$ when E has rank 2. But this is just the projective space $\mathbb{P} \text{Hom}(\mathcal{O}(d), E)$ (regardless of the rank of E).
- (b) In light of (56a) and (54a) our search for a simple E with $\text{Quot } E$ singular leads us to consider the bundle $E := \mathcal{O}_{\mathbb{P}^1}^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. We will study $X := \text{Quot}^{2,-2} E$ for the remainder of this exercise. Write down a point

$$P = (0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0) \in X(\text{Spec } k)$$

such that

$$\dim_k T_P X = h^0(\mathbb{P}^1, S^\vee \otimes Q) = 5$$

and $h^1(\mathbb{P}^1, S^\vee \otimes Q) = 1$.

- (c) Recall the Quot scheme $Q_{2,2} = \text{Quot}^{2,-2} \mathcal{O}_{\mathbb{P}^1}^2$ studied in the previous exercise. If

$$0 \rightarrow S \rightarrow \mathcal{O}_{\mathbb{P}^1}^2 \rightarrow Q \rightarrow 0$$

is a point of $Q_{2,2}$ then show that the splitting type of S is either $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$. The locus $W \subset Q_{2,2}$ where S has the latter splitting type is therefore closed by semicontinuity. Show that there is an isomorphism $W = \text{Sym}^2 \mathbb{P}^1 \times \mathbb{P}^1$ making the diagram

$$\begin{array}{ccc} W & \hookrightarrow & Q_{2,2} \\ & \searrow \pi_1 & \downarrow \det_2 \\ & & \text{Sym}^2 \mathbb{P}^1 \end{array}$$

commute.

(d) Similarly, if

$$0 \rightarrow S \rightarrow \mathcal{O}_{\mathbb{P}^1}^2 \rightarrow Q \rightarrow 0$$

is a point of X then show that the splitting type of S is either $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$, so that the locus $Z \subset X$ where S has the latter splitting type is closed by semicontinuity. Show that $Z \cong \mathbb{P}^1 \times \mathbb{P}^3$. *Hint:* $\mathbb{P}^1 = \mathbb{P} \text{Hom}(\mathcal{O}, \mathcal{O} \oplus \mathcal{O})$ and $\mathbb{P}^3 = \mathbb{P} \text{Hom}(\mathcal{O}(-2), \mathcal{O} \oplus \mathcal{O}(-2))$. On $\mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^1$ we have the “universal map from \mathcal{O} to $\mathcal{O} \oplus \mathcal{O}$ ”

$$\begin{aligned} i : \mathcal{O}(-1, 0, 0) &\rightarrow \mathcal{O}(0, 0, 0)^2 \\ f \otimes s \otimes t &\mapsto sf(t) \end{aligned}$$

pulled back via

$$\pi_{13} : \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^1 \rightarrow \mathbb{P} \text{Hom}(\mathcal{O}, \mathcal{O} \oplus \mathcal{O}) \times \mathbb{P}^1.$$

Let $q : \mathcal{O}(0, 0, 0)^2 \rightarrow Q'$ be the quotient of i . Q' is an invertible sheaf (non-canonically) isomorphic to $\mathcal{O}(1, 0, 0)$. Similarly, we have a universal map

$$\mathcal{O}(0, -1, -2) \rightarrow \mathcal{O}(0, 0, 0) \oplus \mathcal{O}(0, 0, -2)$$

(pulled back via π_{23}) with quotient $r : \mathcal{O}(0, 0, 0) \oplus \mathcal{O}(0, 0, -2) \rightarrow Q$. Show that the composition

$$(r \otimes \text{Id}_{Q'}) \begin{pmatrix} q & 0 \\ 0 & \text{Id}_{Q' \otimes \pi_3^* \mathcal{O}(-2)} \end{pmatrix} : \mathcal{O}(0, 0, 0)^2 \rightarrow Q \otimes Q'$$

determines a morphism $\mathbb{P}^1 \times \mathbb{P}^3 \rightarrow Z$ and that this is in fact an isomorphism.

(e) Show that $X = Q_{2,2} \coprod_W Z$.

(57) **Quot schemes IV.** We noted in (56a) that $\text{Quot}^{1,d} E$ is smooth for any vector bundle E on \mathbb{P}^1 . Let C be a genus 2 curve and let $E = \mathcal{O}_C \oplus \mathcal{O}_C$. We will see in this exercise that $Q := \text{Quot}^{1,-2} E$ is singular. To fix notation, let L be the g_2^1 on C (the only special degree two line bundle) and let $p : C \rightarrow \mathbb{P}^1$ be the map obtained from L , so $L = p^* \mathcal{O}_{\mathbb{P}^1}(1)$.

(a) Show that the map $Q \rightarrow \text{Pic}^2 C$ obtained from the dual of the universal subbundle has fiber \mathbb{P}^1 except over L where the fiber is \mathbb{P}^3 .

(b) Show that the singular locus of Q is a quadric in this \mathbb{P}^3 fiber.

(58) **Quot schemes V.** Let A be a ring. Define A algebras:

$$\begin{aligned} B_1 &= A[\epsilon]/\epsilon^2 \\ B'_1 &= A[\epsilon]/\epsilon^3 \\ B_2 &= A[x]/x^2 \\ B_0 &= B_1 \otimes_A B_2 \\ &= A[\epsilon, x]/(\epsilon^2, x^2) \\ B &= A[\epsilon]/(\epsilon^3, x^2) \\ F_0 &= A[y]/y^2. \end{aligned}$$

Let \underline{B}'_1 be the square zero A algebra extension

$$\begin{array}{ccccccc} \underline{B}'_1 & = & 0 & \longrightarrow & A & \xrightarrow{\epsilon^2} & B'_1 & \longrightarrow & B_1 & \longrightarrow & 0 \\ & & & & & & & & \uparrow & & \\ & & & & & & & & A & & \end{array}$$

and let

$$\begin{array}{ccccccc} \underline{B} & = & 0 & \longrightarrow & A[x]/x^2 & \xrightarrow{\epsilon^2} & B & \xrightarrow{g} & B_0 & \longrightarrow & 0 \\ & & & & & & & & \uparrow & & \\ & & & & & & & & A & & \end{array}$$

be the algebra extension obtained from \underline{B}'_1 by applying $\otimes_A B_2$ (note that everything in sight is flat as an A module).

View F_0 as a B_0 module by restriction of scalars along the surjective A algebra map $f : B_0 \rightarrow F_0$ sending both ϵ and x to y . Let $N_0 \subseteq B_0$ be the kernel of f , so we have an exact sequence

$$0 \longrightarrow N_0 \longrightarrow B_0 \xrightarrow{f} F_0 \longrightarrow 0.$$

Note that F_0 is flat over B_1 (in fact it is free of rank one and naturally identified with B_1 as a B_1 module via the composition of $B_1 \rightarrow B_0$ and f). Explicitly describe the obstruction

$$\omega \in \text{Ext}_{B_0}^1(N_0, A[x]/x^2 \otimes_{B_0} F_0)$$

to lifting f to a surjection $\bar{f} : B \rightarrow F$ with F flat over B'_1 . Note that

$$\begin{aligned} A[x]/x^2 \otimes_{B_0} F_0 &= A \otimes_{B_1} F_0 \\ &= A \otimes_{B_1} B_1 \\ &= A \end{aligned}$$

so we should regard ω as an element of $\text{Ext}_{B_0}^1(N_0, A)$. In particular, show that $\omega \neq 0$.

Hint: Let $L \subset B$ be the kernel of the composition

$$B \xrightarrow{g} B_0 \xrightarrow{f} F_0$$

so that the map $g|_L : L \rightarrow B_0$ clearly lands in (factors through) $N_0 \subset B_0$. Obviously the kernel $A[x]/x^2$ of $B \rightarrow B_0$ is contained in L (via the map $1 \mapsto \epsilon^2$ as in

the diagram defining \underline{B}), and in fact we have an exact sequence as in the top row of the diagram of B modules below:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[x]/x^2 & \xrightarrow{\epsilon^2} & L & \xrightarrow{g|_L} & N_0 & \longrightarrow & 0 \\ & & \downarrow x \mapsto 0 & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & A & \longrightarrow & M & \longrightarrow & N_0 & \longrightarrow & 0 \end{array}$$

The bottom row is defined by pushing out the top row—it defines the obstruction ω (note that it is straightforward to see that M is a B_0 module (i.e. $\epsilon^2 M = 0$), so this B module extension is actually a B_0 module extension). You should find that $M \rightarrow N_0$ admits both a B_1 linear section and a B_2 linear section, but no B_0 linear section.

- (59) **Deformation to the normal cone I.** What is usually called *deformation to the normal cone* in algebraic geometry is the observation that a filtered ring admits a flat deformation over \mathbb{A}^1 to its associated graded.

Definitions. Let A be a ring (or a ring object of a topos). A *grading* on A is a direct sum decomposition $A = \bigoplus_{n \in \mathbb{Z}} A_n$ of the additive abelian group of A such that $A_m A_n \subseteq A_{m+n}$. A *filtration* on A is a decreasing sequence

$$F = (\dots \subseteq F^{n+1} \subseteq F^n \subseteq F^{n-1} \subseteq \dots)$$

of subgroups of $(A, +)$ with $\bigcup_n F^n = A$. A *graded ring* is a ring A equipped with a grading $A = \bigoplus_n A_n$ and a *filtered ring* (A, F) is a ring A equipped with a filtration F . A *morphism of graded rings* $A \rightarrow B$ is a morphism of rings $A \rightarrow B$ which takes A_n into B_n for all n and a *morphism of filtered rings* $(A, F) \rightarrow (B, G)$ is a ring map $A \rightarrow B$ that takes F^n into G^n for all n .

A filtration F is called *bounded above by N* if $F^N = 0$ (hence $F^n = 0$ for all $n \geq N$) and F is called *bounded below by N* if $F^N = B$ (hence $F^n = B$ for all $n \leq N$). A filtration F on A determines a translation invariant topology on A where the F^n are basic open neighborhoods of $0 \in A$.

For example, an ideal $I \subseteq A$ determines a filtration (also denoted I by abuse of notation)

$$I = (\dots \subseteq I^2 \subseteq I \subseteq I^0 = A)$$

bounded below by 0 (so $I^n = A$ for negative n by convention) called the *I -adic filtration*. The standard example of this is where $A \rightarrow B$ is a surjection of rings with kernel I and one considers the I -adic (abuse of notation!) on the various “infinitesimal neighborhoods” A/I^{n+1} (note $A/I = B$ and the filtration in this case is the trivial filtration).

A graded ring $A = \bigoplus_n A_n$ has a natural filtration given by $F^n = \bigoplus_{i \geq n} A_i$. This determines a forgetful functor from graded rings to filtered rings. The filtration T associated to the trivial grading is called the *trivial filtration* and is given by

$$T^n = \begin{cases} B, & n \leq 0 \\ 0, & n > 0. \end{cases}$$

The trivial filtration is not to be confused with the “very trivial filtration” (the adic filtration for the unit ideal) V where $V^n = B$ for all n .

If $A = \bigoplus_n A$ is an N graded ring, then A_0 is a subring of A which is the equalizer of the ring maps $A \rightrightarrows A[N]$, where the first sends a to $a = a \otimes [0]$ and the second sends $a \in A_n$ to $a \otimes [n]$, where the $[n] \in \mathbb{Z}[N]$ is the element of the monoid

algebra $\mathbb{Z}[N]$ corresponding to $n \in N$. (This ring map is the coaction of the Hopf ring $\mathbb{Z}[N]$ on A , where $\mathbb{Z}[N]$ is regarded as a cogroup in rings via the natural cogroup structure on N in monoids and the fact that the monoid algebra functor preserves direct limits, hence cogroups.) Each A_n becomes an A_0 module via the multiplication map $A_0 \times A_n \rightarrow A_n$ from A .

Every ring A has a *trivial grading* where $A = A_0$. The trivial grading is right adjoint to the functor from graded rings to rings that takes $\bigoplus_n A_n$ to A_0 .

Let $A \rightarrow B$ be a morphism of graded rings. Then each B_n becomes an A_0 module by restriction of the B_0 module structure along the map $A_0 \rightarrow B_0$.

The *associated graded* of a filtered ring (A, F) is the graded ring

$$\mathrm{gr}_F A := \bigoplus_n F^n / F^{n+1}.$$

Taking associated graded determines a functor gr from the category of filtered rings to the category of graded rings. For example, the associated graded of a graded ring (given the natural filtration) is itself because

$$\bigoplus_{i \geq n} A_i / \bigoplus_{i \geq n+1} A_i = A_n.$$

For example, the associated graded of the I -adic filtration on the infinitesimal neighborhood rings A/I^{n+1} has associated graded $\mathrm{gr}_I A/I^{n+1} = B \oplus I/I^2 \oplus \cdots \oplus I^n/I^{n+1}$. Each ring extension $\mathrm{gr}_I A/I^{n+1} \rightarrow \mathrm{gr}_I A/I^n$ is the trivial extension by I^n/I^{n+1} , whereas $A/I^{n+1} \rightarrow A/I^n$ is typically a nontrivial ring extension (see Problem 49).

A filtered ring (A, F) also determines a graded $\mathbb{Z}[t]$ subalgebra of $A[t, t^{-1}]$ given by

$$\mathrm{nc}_F A := \bigoplus_n F^n t^{-n}.$$

The graded $\mathbb{Z}[t]$ -algebra $\mathrm{nc}_F B$ is called the *deformation to the normal cone* of (B, F) .

- (a) Consider the ring $\mathbb{Z}[x, y]/(x^2 y^2)$ with the (xy) -adic filtration. Give an explicit presentation² of the $\mathbb{Z}[t]$ algebra $\mathrm{nc}_{(xy)} \mathbb{Z}[x, y]/(x^2 y^2)$ and observe that $\mathrm{gr}_{(xy)} \mathbb{Z}[x, y]/(x^2 y^2) \cong \mathbb{Z}[x, y, z]/(xy, z^2)$. Generalize to give an explicit presentation of $\mathrm{nc}_{(xy)} \mathbb{Z}[x, y]/(x^{n+1} y^{n+1})$ and show that

$$\mathrm{gr}_{(xy)} \mathbb{Z}[x, y]/(x^{n+1} y^{n+1}) \cong \mathbb{Z}[x, y, z]/(xy, z^{n+1}).$$

- (b) Show that the category of filtered rings has coproducts (“tensor products”), and that the forgetful functor $(A, F) \mapsto A$ from filtered rings to rings preserves them. (So one has $(B, G) \otimes_{(A, F)} (C, H) = (B \otimes_A C, G \otimes_F H)$ and it is up to you to define the filtration $G \otimes_F H$ on $B \otimes_A C$.) Prove the analogous statement for graded rings.
- (c) Let (A, F) be a filtered ring. Show that there are cartesian diagrams of graded rings:

$$\begin{array}{ccc} \mathbb{Z}[t] & \longrightarrow & \mathrm{nc}_F A \\ t \rightarrow 0 \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \mathrm{gr}_F A \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{Z}[t] & \longrightarrow & \mathrm{nc}_F A \\ \downarrow & & \downarrow \\ \mathbb{Z}[t, t^{-1}] & \longrightarrow & A[t, t^{-1}] \end{array}$$

²This is a natural exercise since $\mathrm{nc}_F A$ is described in terms of *subrings* and a presentation describes a ring as a *quotient ring*.

(d) If A is a ring with the trivial filtration T , then observe that $\text{nc}_T A = A[t] = \bigoplus_{n \geq 0} At^n$ as a graded ring. A map of filtered rings $(A, T) \rightarrow (B, F)$ will be called a *filtered A -algebra*. Applying the functor nc and the previous observation, the map $(A, T) \rightarrow (B, F)$ induces $A[t] \rightarrow \text{nc}_F B$. Show that this map is flat when $A = k$ is a field. In general, $\text{nc}_F B$ need not be flat over $A[t]$ and in fact $\text{gr}_F B$ need not be flat over A . Consider $A = B = \mathbb{Z}$ with the filtration $F^{\leq 0} = B$, $F^1 = 2\mathbb{Z} \subseteq \mathbb{Z}$, $F^{>1} = 0$. What hypotheses are necessary in general for $\text{nc}_F B$ to be flat over $A[t]$? Probably the right hypothesis is that (B, F) should be *filtered flat* as a filtered A -module (whatever this means).

(60) **Deformation to the normal cone II.** One frequently encountered manifestation of deformation to the normal cone is the fact that a short exact sequence of vector bundles admits a flat deformation over \mathbb{A}^1 to a split short exact sequence. Suppose

$$0 \longrightarrow K \xrightarrow{i} E \xrightarrow{p} Q \longrightarrow 0$$

is a short exact sequence of locally free coherent sheaves on a scheme X . Pulling back to $X \times \mathbb{A}^1 = \text{Spec } \mathcal{O}_X[t]$, we obtain a SES

$$0 \longrightarrow K[t] \xrightarrow{i} E[t] \xrightarrow{p} Q[t] \longrightarrow 0$$

of locally free coherent sheaves on $X \times \mathbb{A}^1$. This extension of $Q[t]$ by $K[t]$ is classified by an element of $\text{Ext}_{X \times \mathbb{A}^1}^1(Q[t], K[t])$. The multiplication by t map

$$t : K[t] \rightarrow K[t]$$

induces a map

$$t : \text{Ext}_{X \times \mathbb{A}^1}^1(Q[t], K[t]) \rightarrow \text{Ext}_{X \times \mathbb{A}^1}^1(Q[t], K[t]).$$

The image of the extension above under this map is the short exact sequence in the bottom row of the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K[t] & \xrightarrow{i} & E[t] & \xrightarrow{p} & Q[t] \longrightarrow 0 \\ & & \downarrow t & & \downarrow & & \parallel \\ 0 & \longrightarrow & K[t] & \longrightarrow & K[t] \oplus_{K[t]} E[t] & \xrightarrow{(0,p)} & Q[t] \longrightarrow 0 \end{array}$$

where the left square is the pushout. Note that all of the sheaves in the bottom row are locally free and

$$K[t] \oplus_{K[t]} E[t] = \frac{K[t] \oplus E[t]}{\{(tk, -ik) : k \in K[t]\}}.$$

Show that pulling back this bottom row along

$$X = X \times \{0\} \hookrightarrow X \times \mathbb{A}^1$$

yields the split sequence

$$0 \longrightarrow K \longrightarrow K \oplus Q \longrightarrow Q \longrightarrow 0,$$

while pulling back along

$$X \times \mathbb{G}_m = \text{Spec } \mathcal{O}_X[t, t^{-1}] \hookrightarrow X \times \mathbb{A}^1$$

yields a sequence isomorphic to π_1^* of the original sequence.

By abuse of notation, let us refer to the ideal of $\text{Sym}^* E$ generated by $K \subseteq E \subseteq \text{Sym}^* E$ simply as K . Note that K is the kernel of the surjection

$$\text{Sym}^* E \rightarrow \text{Sym}^* Q.$$

Endow $\text{Sym}^* E$ with the K -adic filtration. Show that the deformation to the normal cone $\text{nc}_K \text{Sym}^* E$ defined in the previous exercise is isomorphic to

$$\text{Sym}^*(K[t] \oplus_{K[t]} E[t])$$

as a graded $\mathcal{O}_X[t]$ algebra. In particular, the associated graded of the K -adic filtration on $\text{Sym}^* E$ is $\text{Sym}^*(K \oplus Q) = (\text{Sym}^* K) \otimes_{\mathcal{O}_X} (\text{Sym}^* Q)$. Thus we see that the natural flat deformation to a split exact sequence is a special case of the deformation to the normal cone.

(61) **Traceless Hom.** Let A be a ring (object of a topos) and let E be a locally free A module of finite rank r . Note $\text{Hom}(E, E) = E^\vee \otimes E$. Recall the trace map

$$\begin{aligned} \text{tr} : E^\vee \otimes E &\rightarrow A \\ f \otimes e &\mapsto f(e) \end{aligned}$$

and the scalar multiplication map

$$\begin{aligned} \text{id} : A &\rightarrow \text{Hom}(E, E) \\ a &\mapsto (e \mapsto ae). \end{aligned}$$

Show that the trace map is the coefficient of t^{r-1} in the map

$$\begin{aligned} \text{Hom}(E, E) &\mapsto A[t] \\ f &\mapsto \det(f + t \text{Id}) \end{aligned}$$

(here $f + t \text{Id} \in \text{Hom}_{A[t]}(E[t], E[t])$). (The various other coefficients are the so called *characteristic polynomials*; the constant term, for example, is the determinant.)

The *traceless hom* $\text{Hom}(E, E)_0$ (also denoted $(EE^\vee)_0$) is, by definition, the kernel of the trace map, so we have an exact sequence (the *trace sequence*)

$$0 \rightarrow \text{Hom}(E, E)_0 \rightarrow \text{Hom}(E, E) \rightarrow A \rightarrow 0$$

of locally free A modules (it is clear that the trace map is surjective as long as $r > 0$, which we tacitly assume). Show that $\text{tr} \circ \text{id} = r \in A$ so if r is invertible in A , then $r^{-1} \text{id}$ provides a splitting of the trace sequence. If $r = 1$, then note that $E \otimes E^\vee = A$ and the “usual” isomorphism agrees with both the trace map $\text{tr} : E \otimes E^\vee \rightarrow A$ and the scalar multiplication $\text{id} : A \rightarrow E \otimes E^\vee$, so $\text{Hom}(E, E)_0 = 0$.

The first interesting case is when $r = 2$. In this $r = 2$ case, first show that there is a natural isomorphism $\text{Hom}(E, E)_0 = \wedge^2 E \otimes \text{Sym}^2 E^\vee$. Next show that there is a natural isomorphism $\wedge^2 E \otimes E^\vee = E$ and that the trace sequence is identified under these last two isomorphisms with the *sym-skewsym* exact sequence

$$0 \rightarrow \text{Sym}^2 E^\vee \rightarrow E^\vee \otimes E^\vee \rightarrow \wedge^2 E^\vee \rightarrow 0$$

tensoring with $\wedge^2 E$. Finally, show that under this last isomorphism, the map $\text{id} : A \rightarrow E \otimes E^\vee$ is identified with the tensor product of the usual map

$$\begin{aligned} \wedge^2 E^\vee &\rightarrow E^\vee \otimes E^\vee \\ e \wedge f &\mapsto e \otimes f - f \otimes e \end{aligned}$$

and the identity of $\wedge^2 E$. Note that the composition

$$\wedge^2 E^\vee \rightarrow E^\vee \otimes E^\vee \rightarrow \wedge^2 E^\vee$$

is also multiplication by $r = 2$:

$$\begin{aligned} e \wedge f &\mapsto e \otimes f - f \otimes e \\ &\mapsto e \wedge f - f \wedge e \\ &= 2e \wedge f. \end{aligned}$$

It is interesting that the trace sequence (and similarly the sym-skewsym sequence mentioned above) is a short exact sequence naturally associated to E that splits (naturally) when $r \in A$ is invertible. Can you give an example of a rank two locally free module E over a sheaf of rings A where the trace sequence for E does not split?

- (62) **A type of blowup.** Suppose Z is the zero locus of a section s of a rank two vector bundle V on a scheme X . Then $V = \text{Spec } \mathcal{O}_V = \text{Spec } \text{Sym}^* V^\vee$ and the section is Spec of a map of \mathcal{O}_X algebras $s^\sharp : \text{Sym}^* V^\vee \rightarrow \mathcal{O}_X$, which by the adjointness property of Sym^* is uniquely determined by the map of \mathcal{O}_X modules $s_1^\sharp : V^\vee \rightarrow \mathcal{O}_X$ given by the degree one part of s^\sharp . Let I be the ideal sheaf of Z in X , so I is the ideal generated by the image of s_1^\sharp and we have an obvious surjection of graded \mathcal{O}_X algebras

$$\text{Sym}^* V^\vee \rightarrow \bigoplus_{n \geq 0} I^n.$$

Taking Proj , we obtain an embedding $\text{Bl}_Z X \hookrightarrow \mathbb{P}(V)$.

Let us examine this in local coordinates. Suppose we have chosen, locally at least, a trivialization of V so that $\mathcal{O}_V = \mathcal{O}_X[X, Y]$. Then our section s is an \mathcal{O}_X algebra map $s : \mathcal{O}_X[X, Y] \rightarrow \mathcal{O}_X$. Set $x := s(X)$, $y := s(Y)$ so that the ideal I of Z is generated by $x, y \in \mathcal{O}_X$. Show directly that the ideal J generated by $xY - yX \in \mathcal{O}_X[X, Y]$ is independent of the choice of local coordinates, but also give a “global” interpretation of this ideal which makes this obvious.

The ideal J is contained in $\text{Ker } s$, which is the ideal of $s[X] \hookrightarrow V$, but they are not generally equal (for example, s might be the zero section, in which case we will have $J = (0)$). However, if x, y form a regular sequence in \mathcal{O}_X , then prove that $J = \text{Ker } s$.

Since the ideal J is invariant for the \mathbb{G}_m scaling action, it determines a \mathbb{G}_m invariant closed subscheme $B_Z X \hookrightarrow V$. Removing the zero sections and taking quotients by \mathbb{G}_m gives an embedding $(B_Z^* X / \mathbb{G}_m) \hookrightarrow \mathbb{P}(V)$, which coincides with the embedding $\text{Bl}_Z X \hookrightarrow \mathbb{P}(V)$ when $J = \text{Ker } s$.

- (63) **Weil restriction.** Recall that the base change functor

$$\begin{aligned} \mathbf{Sch}/\mathbb{R} &\rightarrow \mathbf{Sch}/\mathbb{C} \\ X &\mapsto X \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C} =: X_{\mathbb{C}} \end{aligned}$$

admits a right adjoint

$$\begin{aligned} \mathbf{r} : \mathbf{Sch}/\mathbb{C} &\rightarrow \mathbf{Sch}/\mathbb{R} \\ X &\mapsto \mathbf{r} X \end{aligned}$$

called the *Weil restriction* (it also admits a left adjoint, given by the obvious forgetful functor). The right adjoint to base change along a flat proper morphism exists in great generality, but in the case of a *finite* flat morphism such as $\text{Spec } \mathbb{C} \rightarrow \text{Spec } \mathbb{R}$ (or $\text{Spec } \mathbb{Z}[t]/t^{n+1} \rightarrow \text{Spec } \mathbb{Z}$, where the Weil restriction is called the space

of n jets) it is easy to construct explicitly. In our case, if X is locally

$$\mathrm{Spec} \mathbb{C}[z_1, \dots, z_n]/(f_1(z_1, \dots, z_n), \dots, f_k(z_1, \dots, z_n)),$$

then we introduce new formal variables $x_1, y_1, \dots, x_n, y_n$ and just formally extract the real and imaginary parts $\Re f_i, \mathrm{Im} f_i \in \mathbb{R}[x_1, y_1, \dots, x_n, y_n]$ of each $f_i(x_1 + iy_1, \dots, x_n + iy_n)$, pretending that the x_i and y_i are real numbers. Then rX will be (locally)

$$\mathrm{Spec} \mathbb{R}[x_1, y_1, \dots, x_n, y_n]/(\Re f_1, \mathrm{Im} f_1, \dots, \Re f_k, \mathrm{Im} f_k).$$

Of course n and k can be infinite, it just makes the notation more confusing. One can argue that these local charts for rX glue to define a scheme rX over $\mathrm{Spec} \mathbb{R}$.

Notice that we could also work over \mathbb{Z} and construct a right adjoint to base change along $\mathbb{Z} \rightarrow \mathbb{Z}[\sqrt{-1}]$ by the same method. Note $\mathrm{Spec} \mathbb{Z}[\sqrt{-1}] = \boldsymbol{\mu}_2$ is a group scheme. The fact that $\mathbb{C} = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}[\sqrt{-1}]$ formally implies that the diagram of functors

$$\begin{array}{ccc} \mathbf{Sch}/\mathbb{Z}[\sqrt{-1}] & \longrightarrow & \mathbf{Sch}/\mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbf{Sch}/\mathbb{C} & \longrightarrow & \mathbf{Sch}/\mathbb{R} \end{array}$$

commutes (or “2-commutes”), where the vertical arrows are base change and the horizontal arrows are Weil restriction. For a $\mathbb{Z}[\sqrt{-1}]$ scheme X , we can therefore write $rX \times_{\mathbb{Z}} \mathbb{R}$ for $(rX) \times_{\mathbb{Z}} \mathbb{R} = r(X \times_{\mathbb{Z}[\sqrt{-1}]} \mathbb{C})$ without ambiguity.

If we obtain $X \times \boldsymbol{\mu}_2 \in \mathbf{Sch}/\mathbb{Z}[\sqrt{-1}]$ by base change, then there is an adjunction morphism $X \rightarrow r(X \times \boldsymbol{\mu}_2)$. By abuse of notation, we will set

$$rX := r(X \times \boldsymbol{\mu}_2)$$

for $X \in \mathbf{Sch}$. In terms of the above local coordinate description, if X is locally

$$\mathrm{Spec} \mathbb{Z}[z_1, \dots, z_n]/(f_1, \dots, f_k),$$

then $X \times \boldsymbol{\mu}_2 = X[\sqrt{-1}]$ is locally $\mathrm{Spec} \mathbb{Z}[\sqrt{-1}][z_1, \dots, z_n]$ and $rX = r(X \times \boldsymbol{\mu}_2)$ is locally

$$\mathrm{Spec} \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]/(\Re f_i, \mathrm{Im} f_i).$$

The adjunction morphism $X \rightarrow rX$ is locally Spec of

$$\begin{aligned} \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n]/(\Re f_i, \mathrm{Im} f_i) &\rightarrow \mathbb{Z}[z_1, \dots, z_n]/(f_i) \\ x_i &\mapsto z_i \\ y_i &\mapsto 0. \end{aligned}$$

Notice that this makes sense because the f_i have “real” (or in this case integer) coefficients, so $\Re f_i(z + i \cdot 0) = f_i(z)$ and $\mathrm{Im} f_i(z + i0) = 0$. The Weil restriction is a right adjoint, so it preserves all inverse limits. In particular, it preserves terminal objects: $r\boldsymbol{\mu}_2 = \mathrm{Spec} \mathbb{Z}$ and products: $r(X \times_{\boldsymbol{\mu}_2} Y) = rX \times rY$.

The adjointness property of r in particular implies that $X(\mathbb{C}) = rX(\mathbb{R})$ as sets, but this set bijection clearly preserves some additional structure. For example, $X(\mathbb{C})$ inherits a Zariski topology and a sheaf of \mathbb{C} algebras $\mathcal{O}_{X(\mathbb{C})}$ from the inclusion $X(\mathbb{C}) \hookrightarrow X$, and $rX(\mathbb{R})$ inherits a Zariski topology and a sheaf $\mathcal{O}_{rX(\mathbb{R})}$ of \mathbb{R} algebras from rX . The Zariski topology on $rX(\mathbb{R})$ lies between the Zariski topology on $X(\mathbb{C})$ and the “classical” (i.e. metric) topology on $X(\mathbb{R})$; this latter topology coincides with the metric topology on $X(\mathbb{C})$ in the sense that the set

bijection $X(\mathbb{C}) = rX(\mathbb{R})$ is a homeomorphism in the metric topology. Similarly, $\mathcal{O}_{rX(\mathbb{R})} \otimes_{\mathbb{R}} \mathbb{C}$ lies between (in some sense) $\mathcal{O}_{X(\mathbb{C})}$ and the sheaf of continuous \mathbb{C} valued functions on $X(\mathbb{C})$ (using the metric topologies for the notion of “continuous”).

The Weil restriction preserves étale morphisms, affine morphisms, and smooth morphisms and doubles dimension.

- (a) Show directly from the picture of a “standard étale morphism” that Weil restriction preserves étale morphisms.
- (b) Conclude from the aforementioned properties of Weil restriction that it *cannot* preserve proper morphisms. *Hint:* A smooth proper curve can be covered by two (connected) affines, but a smooth proper surface cannot. (Why?)
- (c) Taking \mathbb{R} points of the adjunction morphism $\mathbb{P}_{\mathbb{R}}^1 \rightarrow r\mathbb{P}_{\mathbb{C}}^1$, we obtain a map: $S^1 \cong \mathbb{P}_{\mathbb{R}}^1 \hookrightarrow (r\mathbb{P}_{\mathbb{C}}^1)(\mathbb{R}) = \mathbb{P}_{\mathbb{C}}^1(\mathbb{C}) = S^2$. Show that this copy of S^1 in S^2 is an equator in the “usual” metric on $\mathbb{P}^1(\mathbb{C}) = S^2$ as described, for example, in: H. Weyl, *On unitary metrics in projective space*. Math. Ann. 40 (1939) 141-148.
- (d) Show that the Weil restriction of the node $\text{Spec } \mathbb{C}[z_1, z_2]/z_1z_2$ is

$$\text{Spec } \mathbb{R}[x_1, x_2, y_1, y_2]/(x_1x_2 - y_1y_2, x_1y_2 + x_2y_1),$$

which is a cone on a smooth variety $X \subset \mathbb{P}_{\mathbb{R}}^3$ whose \mathbb{R} points $X(\mathbb{R})$ form two disjoint, unlinked circles. This is as expected: the node is topologically a cone on two circles.

- (64) **Weil restriction of \mathbb{G}_m .** The Weil restriction $r\mathbb{G}_m$ is described extensively in Exercise 31. The Weil restriction $r\mathbb{G}_m := r(\mathbb{G}_m \times \mu_2) \in \mathbf{Sch}/\mathbb{Z}$ of (the base change of) the multiplicative group $\mathbb{G}_m \times \mu_2 \in \mathbf{Sch}/\mu_2$ is isomorphic to the twisted quotient $\mathbb{G}_m \times_{\mu_2} U(1)$. We encounter the usual notational clash here: the notation \times_{μ_2} here does not denote a fibered product (what would the maps to μ_2 even be?), but rather the group quotient by the diagonally embedded subgroup $\mu_2 \hookrightarrow \mathbb{G}_m \times U(1)$. We will set

$$r\mathbb{G}_m := r(\mathbb{G}_m \times_{\mathbb{Z}} \mathbb{Z}[\sqrt{-1}])$$

throughout by slight abuse of notation. Recall $U(1) := \text{Spec } \mathbb{Z}[x, y]/\langle x^2 + y^2 - 1 \rangle$, so

$$U(1) \times \mathbb{G}_m = \text{Spec } \mathbb{Z}[x, y, t, t^{-1}]/\langle x^2 + y^2 - 1 \rangle.$$

The μ_2 action is given by the $\mathbb{Z}/2\mathbb{Z}$ grading where x, y, z all have grading one.

If one follows the recipe for computing the Weil restriction of an affine scheme given above, then the following description of $r\mathbb{G}_m$ is obtained:

$$r\mathbb{G}_m = \text{Spec } \mathbb{Z}[x, y]_{x^2+y^2}$$

with the multiplication, identity, and inverse maps given by Spec of the maps

$$\begin{aligned}
\mathbb{Z}[x, y]_{x^2+y^2} &\rightarrow \mathbb{Z}[x, y]_{x^2+y^2} \otimes \mathbb{Z}[x, y]_{x^2+y^2} \\
x &\mapsto x \otimes x - y \otimes y \\
y &\mapsto x \otimes y + y \otimes x \\
\mathbb{Z}[x, y]_{x^2+y^2} &\rightarrow \mathbb{Z}[x, y]_{x^2+y^2} \\
x &\mapsto \frac{x}{x^2+y^2} \\
y &\mapsto \frac{-y}{x^2+y^2} \\
\mathbb{Z}[x, y]_{x^2+y^2} &\rightarrow \mathbb{Z} \\
x &\mapsto 1 \\
y &\mapsto 0.
\end{aligned}$$

The isomorphism of group schemes $\mathrm{r}\mathbb{G}_m = \mathbb{G}_m \times_{\mu_2} U(1)$ is obtained as Spec of the Hopf algebra isomorphism

$$\begin{aligned}
(\mathbb{Z}[x, y, t, t^{-1}]/\langle x^2 + y^2 - 1 \rangle)_0 &\rightarrow \mathbb{Z}[x, y]_{x^2+y^2} \\
ty &\mapsto y \\
tx &\mapsto x \\
x^2 &\mapsto \frac{x^2}{x^2+y^2} \\
y^2 &\mapsto \frac{y^2}{x^2+y^2} \\
t^2 &\mapsto x^2 + y^2.
\end{aligned}$$

The subscript 0 here denotes the degree zero part of the $\mathbb{Z}/2\mathbb{Z}$ grading on the coordinate ring of $U(1) \times \mathbb{G}_m$ corresponding to the μ_2 action.

Applying the Weil restriction functor to the action $a : \mathbb{G}_m \times \mathbb{A}_z^1 \rightarrow \mathbb{A}_z^1$ yields a map $\mathrm{r}a : (\mathrm{r}\mathbb{G}_m) \times \mathbb{A}_{x,y}^2$ which is clearly linear. In fact, it is inherited via the embedding $(\mathrm{r}\mathbb{G}_m) \hookrightarrow \mathrm{GL}_2$ given by Spec of

$$\begin{aligned}
\mathrm{Spec} \mathbb{Z}[a, b, c, d]_{ad-bc} &\rightarrow \mathrm{Spec} \mathbb{Z}[x, y]_{x^2+y^2} \\
a &\mapsto x \\
b &\mapsto -y \\
c &\mapsto y \\
d &\mapsto x \\
ad - bc &\mapsto x^2 + y^2.
\end{aligned}$$

That is, for any scheme X , we have

$$\mathrm{r}\mathbb{G}_m(X) = \left\{ \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \in \mathrm{GL}_2(X) \right\}.$$

The adjunction morphism $\mathbb{G}_m \rightarrow \mathrm{r}\mathbb{G}_m$ is given by

$$s \mapsto \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}.$$

The action of $\mathrm{r}\mathbb{G}_m < \mathrm{GL}_2$ on $\mathbb{A}_{x,y}^2$ preserves, up to multiplication by a “sum of squares unit”, the form $\mathbb{A}^2 \times \mathbb{A}^2 \rightarrow \mathbb{A}^2$ given by

$$(x_1, y_1), (x_2, y_2) \mapsto (x_1x_2 + y_1y_2, x_2y_1 - x_1y_2).$$

There is a natural μ_2 action on $\mathrm{r}\mathbb{G}_m$ inherited from the action of μ_2 on either factor of the twisted quotient $U(1) \times_{\mu_2} \mathbb{G}_m$ (the two actions are identified on the twisted quotient, but neither is trivial because the twisted quotient is formed via the diagonal action). This action corresponds to the $\mathbb{Z}/2\mathbb{Z}$ grading on the ring

$$(\mathbb{Z}[x, y, t, t^{-1}]/(x^2 + y^2 - 1))_0$$

where tx, ty have grading 1 and x^2, y^2, t^2 have grading zero. Or equivalently, the $\mathbb{Z}/2\mathbb{Z}$ grading on the ring $\mathbb{Z}[x, y]_{x^2+y^2}$ where x, y have grading one. Actually, we will be more concerned with the μ_2 action on $\mathrm{r}\mathbb{G}_m$ obtained from the $\mathbb{Z}/2\mathbb{Z}$ grading where ty has grading 1 and tx, x^2, y^2, t^2 have grading zero, or equivalently the $\mathbb{Z}/2\mathbb{Z}$ grading on $\mathbb{Z}[x, y]_{x^2+y^2}$ where y has grading 1 and x has grading zero. This latter action is given for a scheme X by

$$\begin{aligned} \mu_2(X) \times \mathrm{r}\mathbb{G}_m(X) &\rightarrow \mathrm{r}\mathbb{G}_m(X) \\ \left(v, \begin{pmatrix} s & -t \\ t & s \end{pmatrix} \right) &\mapsto \begin{pmatrix} s & -vt \\ vt & s \end{pmatrix}. \end{aligned}$$

The fixed subgroup of this latter μ_2 action is the adjunction subgroup $\mathbb{G}_m \hookrightarrow \mathrm{r}\mathbb{G}_m$. Indeed, under the identification $\mathrm{r}\mathbb{G}_m = \mathbb{Z}[x, y]_{x^2+y^2}$, the adjunction morphism is Spec of

$$\begin{aligned} \mathbb{Z}[x, y]_{x^2+y^2} &\rightarrow \mathbb{Z}[t, t^{-1}] \\ x &\mapsto t \\ y &\mapsto 0. \end{aligned}$$

This map induces an isomorphism

$$\begin{aligned} \mathbb{Z}[x, y]_{x^2+y^2}/y &= (\mathbb{Z}[x, y]/y)_{x^2} \\ &= \mathbb{Z}[x]_x \\ &= \mathbb{Z}[t, t^{-1}]. \end{aligned}$$

Note that $(\mathrm{r}\mathbb{G}_m)^{\mu_2} = \mathrm{Spec} \mathbb{Z}[x, y]_{x^2+y^2}/y$ because the fixed locus of a diagonalizable group action is the closed subscheme cut out by the ideal generated by the elements with nonzero grading.