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Matt,

I sorted out this business with the Gromov-Witten invariants of the cubic surface, and some of their relations to the classical geometry as we were discussing. It occurred to me that there is some discussion of this, with a reference, in *Mirror Symmetry and Algebraic Geometry* by Cox and Katz, but I don't have my math books with me, so you have to live with this ad hoc exposition. I don't have anything more to say about the computation of the number of singular plane sections through two generic points via degenerations, so you'll have to wait on that.

Here are some enumerative questions we can answer:

Question 1. Let $P, Q \in S_3 \subseteq \mathbb{P}^3$ be two generic points. How many planes in $\mathbb{P}^3$ containing P, Q have singular intersection with S_3? Answer: 12.

Question 2. How many planes in $\mathbb{P}^3$ intersect S_3 in a triangle? Answer: 45. Here and throughout, we agree that a *triangle* is a configuration of three $\mathbb{P}^1$'s such that any two intersect in a point. A triangle has (arithmetic) genus one.

Question 3. Let $P \in S_3$ be a generic point. How many conic curves in $\mathbb{P}^3$ are contained in S_3 and contain P? Answer: 27.

Question 4. Let $P, Q \in S_3 \subseteq \mathbb{P}^3$ be two generic points. How many degree 3 rational curves in $\mathbb{P}^3$ are contained in S_3 and meet P, Q? Answer: 84.

So that this is comprehensible, let's agree to the following notation. Let $\iota : S_3 \hookrightarrow \mathbb{P}^3$ be a (smooth) cubic hypersurface. Abstractly, S_3 is isomorphic to the blowup of $\mathbb{P}^2$ at 6 points $P_1, \dots, P_6$ (no three colinear, and not all contained in a conic), so let $E_i \cong \mathbb{P}^1$ ($i = 1, \dots, 6$) be the exceptional divisors (use the same notation for their Chow classes in $A^1(S_3) \cong \mathbb{Z}^7$), and let $H \in A^1(S_3)$ be the pullback of the class of a line in $\mathbb{P}^2$. By abuse of notation, we may sometimes think of H as being the $\mathbb{P}^1 \subset S_3$ obtained as the proper transform of a line in $\mathbb{P}^2$ under the blowup map $S_3 \to \mathbb{P}^2$, since this is a representative of the Chow class we called H. The formula for Chern classes of the tangent bundle of a blowup gives

$$c_1(TS_3) = 3H - E_1 - \cdots - E_6.$$

The basis $H, E_1, \dots, E_6$ for $A^1(S_3)$ puts its intersection form in the "standard form" $\mathrm{diag}(1, -1, \dots, -1)$, so the dual (ordered) basis is just $H, -E_1, \dots, -E_6$. The embedding ι of S_3 into $\mathbb{P}^3$ is given by the very ample line bundle

$$\iota^* \mathscr{O}_{\mathbb{P}^3}(1) = \mathscr{O}_{S_3}(3H - E_1 - \cdots - E_6),$$

so the E_i are embedded as lines in $\mathbb{P}^3$ and H is embedded as a rational normal curve. Note that the (very) ample bundle $\mathscr{O}_{\mathbb{P}^3}(1)|_{S_3}$ is nothing but the anti-canonical bundle of S_3, so S_3 is Fano, as we all know, and its embedding as the degree 3 hypersurface is the anti-canonical embedding.

For a smooth projective variety X, an effective curve class β, a genus $g \geq 0$, and cohomology/Chow classes $\alpha_1, \ldots, \alpha_n$ on X, write

$$\langle \alpha_1, \ldots, \alpha_n \rangle^X_{g,\beta} := \int_{[\overline{M}_{g,n}(X,\beta)]^{\mathrm{vir}}} e_1^*\alpha_1 \cdots e_n^*\alpha_n$$

for the corresponding Gromov-Witten invariant. This is clearly multilinear in the α_i:

$$\langle \underbrace{\_\,, \ldots, \,\_}_{n} \rangle_{g,\beta} : A^*(X)^{\otimes n} \to \mathbb{Q}.$$

If $g = 0$, we also drop it from the notation, so if there is only one subscript it indicates degree and genus is assumed zero. If $X = S_3$, we'll drop the superscript X from the notation. Define

$$\langle \alpha_1, \ldots, \alpha_n \rangle_d := \sum_{\iota_*\beta = d \in A_1(\mathbb{P}^3) = \mathbb{Z}} \langle \alpha_1, \ldots, \alpha_n \rangle_d.$$

There is no worry about "infinite sums" or any of that nonsense because the stable map spaces are projective degree-by-degree (relative to any very ample line bundle), as one proves as part of their construction. Said another way: you construct $\overline{M}_{g,n}(X,d)$ as a closed subscheme of $\overline{M}_{g,n}(\mathbb{P}^N, d)$, and the latter is projective. The cutting up into finer notions of "degree" is ex-posto-facto. As long as your notion of "degree" is constant in families of stable maps, you can't have infinitely many (effective) "degrees" mapping to the same degree (relative to the ample line bundle) simply for compactness reasons (thus you probably should use algebraic instead of rational equivalence in your Chow groups). In fact, the best notion of "degree" of an effective curve is: an equivalence class of effective curves where two such are equivalent if they are represented by two points in the same connected component of a stable map space. In any case, we will see explicitly in the case of S_3 that there are only finitely many effective classes in $A_1(S_3) \cong \mathbb{Z}^7$ of a given degree d in $\mathbb{P}^3$.

The class $[\overline{M}_{g,n}(S_3, d)]^{\mathrm{vir}}$ is in $A_e(\overline{M}_{g,n}(S_3, d))$, where

$$\begin{aligned} e &= (\dim S_3 - 3)(1-g) + n + \int_d c_1(TS_3) \\ &= g - 1 + n + d \end{aligned}$$

is the expected dimension, calculated by our Chern formula or the usual adjunction argument (in particular, note that $\int_\beta c_1(TS_3)$ only depends on $\iota_*\beta \in A_1(\mathbb{P}^3)$ because $c_1(TS_3)$ is pulled back from $A^1(\mathbb{P}^3)$). Since $A^*(S_3)$ is generated by divisor classes, Kontsevich's Reconstruction Theorem implies that all genus zero GW invariants of S_3 are uniquely determined by the invariants with at most two cohomology insertions, together with the WDVV equations. For reasons of dimension, the Point Mapping Axiom (a GW invariant in non-zero degree or genus or with more than three insertions is zero if one of the cohomology insertions is trivial), and the Divisor Axiom, the full genus zero theory of S_3 is determined by the invariants: $\langle\rangle_L$, $\langle pt\rangle_C$, and $\langle pt, pt\rangle_T$, where $L \in A_1(S_3)$ ranges over effective classes having degree one in $\mathbb{P}^3$ (i.e. lines), $C \in A_1(S_3)$ ranges over effective classes having degree two in $\mathbb{P}^3$ (conics and their degenerate versions), and $T \in A_1(S_3)$ ranges over effective classes having degree three in $\mathbb{P}^3$. It turns out that all of these invariants are determined by WDVV and the geometry of the configuration of lines in S_3.

Recall where we find the 27 lines on S_3: we have the six E_i, the (proper transforms of the) 15 lines L_{ij} between P_i and P_j in $\mathbb{P}^2$, and the (proper transforms of the) 6 conics $C_i \subset \mathbb{P}^2$, uniquely defined by the fact that all $P_j \in C_i$ for all $j \neq i$. (Any 5 points in $\mathbb{P}^2$, no three colinear, are on a unique conic, as we recall from the $d = 2$ case of Kontsevich's formula for the number of degree d rational plane curves through $3d - 1$ (generic) points in $\mathbb{P}^2$.) In our basis we have

$$L_{ij} = H - E_i - E_j$$
$$C_i = 2H - \sum_{j \neq i} E_j.$$

It is clear that these *are* lines in $\mathbb{P}^3$, since we can calculate that each has degree one. For example,

$$\begin{aligned} c_1(\mathscr{O}_{\mathbb{P}^3}(1)|_{S_3}) \cdot C_i &= (3H - E_1 - \cdots - E_6) \cdot (2H - \sum_{j \neq i} E_j) \\ &= 6H \cdot H + \sum_{j \neq i} E_j \cdot E_j \\ &= 1. \end{aligned}$$

It probably does not require much more effort to see that these are the only lines, as we know. I think what you do is try to bound the coefficient of H in the Chow class of a line by using the intersection form, then check by inspection that these are the only lines. In what follows we will use L to refer generically to one of these line classes.

In the GW notation, we have $\langle\rangle_1 = 27$ (no cohomology insertions); more precisely, $\langle\rangle_L = 1$ for each of the 27 line classes $L \in A_1(S_3)$. Note that each line has a different Chow class, so we see that there are exactly 27 effective classes in $A_1(S_3)$ having degree one in $\mathbb{P}^3$. The fact that each line meets 10 others is just a simple observation about this description of the lines:

E_i meets the five L_{ij} where $j \neq i$ and the five C_j where $j \neq i$.

C_i meets the five L_{ij} where $j \neq i$ and the five E_j where $j \neq i$.

L_{ij} meets E_i, E_j, C_i, C_j, and the six L_{kl} where $\{k, l\} \cap \{i, j\} = \emptyset$.

Musical interlude:

> Let it rain, let it pour,
> Let it rain a whole lot more,
> 'Cause I got them deep river blues.
> Let the rain drive right on,
> Let the waves sweep along,
> 'Cause I got them deep river blues.

We can classify the triangles as follows:

Lemma 5. *For each line L in S_3 there are exactly five triangles $\{L, L', L''\}$ containing L. Indeed, the ten lines meeting L break into five pairs where the two lines in each pair intersect. Any two intersecting lines are contained in a unique triangle. The Chow class $C(L) := L' + L''$ is independent of the triangle containing L, and is distinct for each L. The Chow class $L + L' + L''$ is $3H - E_1 - \cdots - E_6 = \mathscr{O}_{\mathbb{P}^3}(1)|_{S_3}$ for any triangle $\{L, L', L''\}$. There are 45 total triangles.*

Proof. This is just a matter of inspecting the list of lines and the list of lines meeting a given line. We have

$$\begin{aligned} C(L_{ij}) &= 2H - \sum_{k \neq i,j} E_k \\ C(E_i) &= 3H - E_1 - \cdots - 2E_i - \cdots - E_6 \\ C(C_j) &= H - E_j. \end{aligned}$$

The 45 triangles are the 30 triangles $\{C_i, E_j, L_{ij}\}$ (over distinct $i, j \in \{1, \ldots, 6\}$) and the 15 triangles $\{L_{ij}, L_{kl}, L_{mn}\}$ (over partitions of $\{1, \ldots, 6\}$ into three sets of size two). □

The lemma also solves the enumerative Question 2, since a triangle $\{L, L', L''\}$ in S_3 determines a unique plane in $\mathbb{P}^3$ whose intersection with S_3 is $\{L, L', L''\}$: namely, $\mathrm{Span}(L, L', L'')$.

It is probably worth emphasizing at this point that the 27 lines intersect each other in nodes, meaning that no more than two lines intersect at any given point. This is clear from the description of the lines.

We now explain the answer to Question 3.

Proposition 6. $\langle pt \rangle_2 = 27$.

Proof. The invariant gives the number of conics in $\mathbb{P}^3$ contained in S_3 and containing a generic point P of S_3. Since P is generic, it is not on any line, so these must be genuine conics, so a conic incident to P spans a plane in $\mathbb{P}^3$; this plane must also intersect S_3 in a line ("residually") by degree considerations. Conversely, given one of the 27 lines L, look at the plane in $\mathbb{P}^3$ spanned by L and P. It must intersect S_3 residually in a conic for degree reasons; this conic must contain P because P is contained in the plane but not the line. We have described a bijection between the set of lines in S_3 and the set of conics in S_3 containing P. □

The statement of the proposition is not perfectly precise, since we would like this answer Chow class by Chow class, not only degree by degree. We should start by thinking about the set of effective Chow classes in $A_1(S_3)$ having degree 2 in $\mathbb{P}^3$. Certainly the classes $2L$ are such, as we see by double covering one of the the 27 lines L. Another source of such classes is to add two line classes where the (unique!) effective representatives intersect ("effective" here is "connected effective"): these classes are exactly the classes $C(L)$ of Lemma 5.

In fact, the proof of Proposition 6 shows that $\langle pt \rangle_{C(L)} = 1$ for each line L. To see this, recall that for generic P we found the 27 conics in S_3 meeting P by looking at the conic residual to $\mathrm{Span}(L,P) \cong \mathbb{P}^2$ in S_3. I claim this conic has Chow class $C(L)$. To see this, think about what happens when P degenerates to a generic point of a line L' incident to L. The residual conic must degenerate to (the residual part of) $\mathrm{Span}(L,L') \cap S_3$. Since $\mathrm{Span}(L,L') \cap S_3$ already contains two lines and is degree three in the plane $\mathrm{Span}(L,L')$, it must be a triangle of lines $\{L,L',L''\}$, so the residual part is the pair of intersecting lines L', L'' and the Chow class of our conic is $L' + L'' = C(L)$ (c.f. Lemma 5). Probably, with a little more effort, one could show that

$$\overline{M}_{0,1}(S_3, C(L)) \cong \{(P,V) \in S_3 \times \mathbb{P}^{3*} : \mathrm{Span}(L,P) \subseteq V\},$$

with the evaluation map e_1 identified with projection on the first factor. If $L \cap L' = \{P\}$, then

$$(P, \mathrm{Span}(L,L')) \in \{(P,V) \in S_3 \times \mathbb{P}^{3*} : \mathrm{Span}(L,P) \subseteq V\}$$

will correspond to the inclusion of $L' \cup L''$ where $\{L,L',L''\}$ is the unique triangle containing L and L'. If Q is the intersection point of L' and L'' in this situation, then

$$(Q, \mathrm{Span}(L',L'')) \in \{(P,V) \in S_3 \times \mathbb{P}^{3*} : \mathrm{Span}(L,P) \subseteq V\}$$

will correspond to the stable map whose domain is a chain of three $\mathbb{P}^1$'s, with the marking on the middle $\mathbb{P}^1$ and the stable map will map the "left" $\mathbb{P}^1$ isomorphically onto L', collapse the middle $\mathbb{P}^1$ to Q, and map the "right" $\mathbb{P}^1$ isomorphically onto L''.

We have proved that the effective Chow classes of degree two are exactly the 27 classes $2L$ and the 27 classes $C(L)$. Indeed, if an effective conic is not a double cover of a line, then it spans a plane and it is in class $C(L)$ where L is the line residual to the conic in this plane.

Lemma 7. $\langle pt \rangle_{2L} = 0$.

Proof. The lines in S_3 are rigid (that is, (-1)-curves), so they can be contracted in S_3 by Castelnuovo, so any curve of degree $2L$ must be a double cover of L, hence such a curve does not meet a generic point of S_3. □

Another way of saying this is that the evaluation map $e_1 : \overline{M}_{0,1}(S_3, 2L) \to S_3$ is constrained to land in $L \subseteq S_3$, so

$$e_1^* : H^*(S_3) \to H^*(\overline{M}_{0,1}(S_3, 2L))$$

factors through the restriction map $H^*(S_3) \to H^*(L)$, hence it kills the point class for dimension reasons. Note that $\overline{M}_{0,1}(S_3, 2L) \cong \overline{M}_{0,1}(\mathbb{P}^1, 2)$ has dimension three, but expected dimension 2, with the excess dimension accounted for by the fact that $h^1(C, f^* N_{L/S_3}) = 1$ for any degree two map $f : C \to L$ with C of genus zero.

We discussed at some point that $\overline{M}_{0,0}(\mathbb{P}^1, 2)$ is a $\mathbb{Z}_2$-gerbe over its coarse moduli space $\mathrm{Sym}^2 \mathbb{P}^1 \cong \mathbb{P}^2$ (given by the positions of the ramification points; the domain curve has two $\mathbb{P}^1$ components when the ramification points come together). It is the nontrivial $\mathbb{Z}_2$-gerbe due to the possibility of monodromy around the diagonal.

Give me back my old boat,
I'm gonna sail if she'll float,
'Cause I got them deep river blues,
I'm goin' back to Muscle Shoals,
Times are better there I'm told,
Cause I got them deep river blues.

Doc Watson

Deep River Blues

Second Proof of Proposition 6. Let me prove my claim that the GW invariant $\langle pt\rangle_{C(L)}$ is determined already from WDVV and the geometry of the lines. I think this is very beautiful.

The following lemma is useful for dealing with the diagonal constraint in the WDVV formula:

Lemma 8. *Let X be a smooth complex surface, $T_1, \dots, T_N$ a basis for $H^2(X)$ with dual basis $T^1, \dots, T^N$. Then for $\beta, \gamma \in A_1(X)$, we have*

$$\sum_{a=1}^{N} \langle \alpha_1, \dots, \alpha_n, T^a\rangle_\beta \langle \alpha'_1, \dots, \alpha'_{n'}, T_a\rangle_\gamma = (\beta\cdot\gamma)\langle \alpha_1, \dots, \alpha_n\rangle_\beta \langle \alpha'_1, \dots, \alpha'_{n'}\rangle_\gamma$$

Proof. This follows from the Divisor Axiom and the fact that

$$\sum_a (T_a\cdot\beta)(T^a\cdot\gamma) = (\beta\cdot\gamma),$$

since both sides give the intersection number of $\beta\times\gamma$ and the diagonal in $X\times X$ by standard facts. □

Choose a triangle $\{L, L', L''\}$ containing L (there are five such by Lemma 5), so that $C(L) = L' + L''$. Apply WDVV using the rational equivalence

$$D(L'', L'|L, L) \sim D(L'', L|L', L)$$

in degree $C(L)$. That is, integrate

$$e_1^* L'' \cdot e_2^* L' \cdot e_3^* L \cdot e_4^* L$$

over the divisors $D(1,2|3,4), D(1,3|2,4)$ in $\overline{M}_{0,4}(S_3, 3)$ parameterizing maps where the domain curves are nodal and the marked points are split across components as indicated. The answer is the same with either divisor since the two divisors are rationally equivalent, as they are pulled back from the equivalence of $0, \infty \in \mathbb{P}^1$ under the stabilization morphism

$$\overline{M}_{0,4}(S_3, C(L)) \to \overline{M}_{0,4} \cong \mathbb{P}^1.$$

Each divisor in question is isomorphic to

$$\coprod_{M+M'=C(L)} \overline{M}_{0,3}(S_3, M) \times_{S_3} \overline{M}_{0,3}(S_3, M'),$$

with the fibered product taken via evaluation at the third marked points.

Looking only at the terms of maximal degree, we have

$$\begin{aligned} & \langle L'', L', pt\rangle_{C(L)}\langle L, L, 1\rangle_0 + \langle L'', L', 1\rangle_0\langle L, L, pt\rangle_{C(L)} \\ = \ & \langle L'', L, pt\rangle_{C(L)}\langle L', L, 1\rangle_0 + \langle L'', L, 1\rangle_0\langle L', L, pt\rangle_{C(L)} \end{aligned}$$

(modulo terms expressed using GW invariants of lower degree). Now, with the exception of the second of the four terms, all these terms vanish by the Divisor Axiom and the fact that $L' \cdot C(L) = L' \cdot (L' + L'') = -1 + 1 = 0$ (and similarly $L'' \cdot C(L) = 0$), so we will be able to solve for

$$\langle L'', L', 1\rangle_0\langle L, L, pt\rangle_{C(L)} = (L \cdot C(L))^2\langle pt\rangle_{C(L)} = 4\langle pt\rangle_{C(L)}.$$

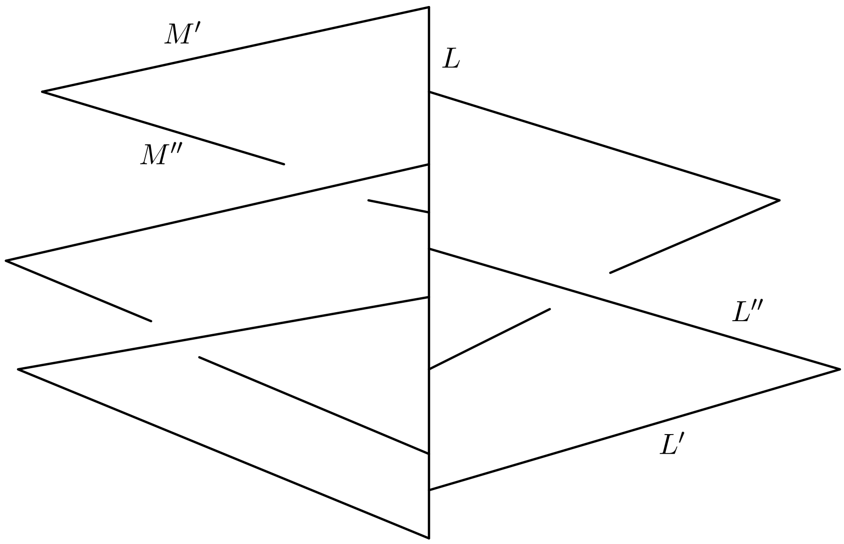

Figure 1: A line $L \subset S_3$ and the five triangles containing it. Both L' and L'' are disjoint from both M' and M'' in any splitting $C(L) = M' + M''$ of $C(L) = L' + L''$ as a sum of two lines, unless $(L', L'') = (M', M'')$ or $(L'', L') = (M', M'')$.

The lower degree terms are where the degree $C(L)$ splits as a sum of lines $M' + M''$. The only way this can happen is if $\{M', M'', L\}$ is a triangle, which means that, after possibly switching M' and M'' we have $M' = L'$ and $M'' = L''$ *or* M', M'' are both disjoint from L' and L'' (see Figure 1). The Divisor Axiom implies that terms of the second type do not contribute, so after simplifying a bit we find:

$$
\begin{aligned}
& 4\langle pt\rangle_{C(L)} \\
& +\sum_{a=1}^{7}\langle L', L'', T^a\rangle_{L'}\langle L, L, T_a\rangle_{L''} \\
& +\sum_{a=1}^{7}\langle L', L'', T^a\rangle_{L'}\langle L, L, T_a\rangle_{L''} \\
= & \sum_{a=1}^{7}\langle L'', L, T^a\rangle_{L'}\langle L', L, T_a\rangle_{L''} \\
& +\sum_{a=1}^{7}\langle L', L'', T^a\rangle_{L''}\langle L, L, T_a\rangle_{L'}.
\end{aligned}
$$

Simplifying this a bit with Lemma 8 and the Divisor Axiom, we find

$$
\begin{aligned}
& 4\langle pt\rangle_{C(L)} \\
& +(L'\cdot L'')(L'\cdot L')(L''\cdot L')(L\cdot L'')(L\cdot L'') \\
& +(L'\cdot L'')(L'\cdot L')(L''\cdot L')(L\cdot L'')(L\cdot L'') \\
= & (L'\cdot L'')(L''\cdot L')(L\cdot L')(L'\cdot L'')(L\cdot L'') \\
& +(L''\cdot L')(L'\cdot L'')(L''\cdot L'')(L\cdot L')(L\cdot L').
\end{aligned}
$$

All the intersection numbers in question are ± 1, and we find $4\langle pt\rangle_{C(L)} - 2 = 2$, which gives the desired result $\langle pt\rangle_{C(L)} = 1$.

Whether you find that beautiful or not is probably a matter of taste.... Beauty is in the eye of the beholder, as they say.

Degree 3 Invariants
Or: *How I learned to stop worrying and love the lines.*

> If my boat sinks with me,
> I'll go down, don't you see,
> 'Cause I got them deep river blues,
> Now I'm gonna say goodbye,
> And if I sink, just let me die,
> 'Cause I got them deep river blues.
>
> Doc Watson
>
> Deep River Blues

In this section, we will answer Question 4. Mostly we will use GW techniques, but we will explain the invariants we compute with WDVV in terms of classical geometry.

Proposition 9. $\langle pt, pt\rangle_H = 1$.

Proof. The invariant counts the number of rational normal curves in $\mathbb{P}^3$ contained in S_3 (in class H) and passing through two general points P, Q of S_3. Since P, Q are general, the line in $\mathbb{P}^2$ between there images under the blowup map $S_3 \to \mathbb{P}^2$ is disjoint from $P_1, \ldots, P_6$, so this line may be viewed as a smooth rational curve in S_3 between P and Q of class H. (It is a rational normal curve in $\mathbb{P}^3$ since H has degree three.) Conversely, the image of a curve in S_3 of class H under the blowup map must be a line in $\mathbb{P}^2$, so this is the unique such curve. □

Proposition 10. *For distinct* $i, j, k \in \{1, \ldots, 6\}$, $\langle pt, pt \rangle_{2H-E_i-E_j-E_k} = 1$.

Proof. (Deland) The invariant counts the number of rational normal curves in $\mathbb{P}^3$ contained in S_3 (in class $2H - E_i - E_j - E_k$) and passing through two general points P, Q of S_3. Since P, Q are general, there is a unique conic in $\mathbb{P}^2$ containing P_i, P_j, P_k and the images of P, Q under the blowup map $S_3 \to \mathbb{P}^2$. Taking the proper transform of this conic in S_3 gives a smooth rational curve in S_3 between P and Q of class $2H - E_i - E_j - E_k$. It is a rational normal curve in $\mathbb{P}^3$ since $2H - E_i - E_j - E_k$ has degree three. Conversely, the image of a curve in S_3 of class $2H - E_i - E_j - E_k$ under the blowup map must be a conic in $\mathbb{P}^2$, and there is a unique conic through 5 general points, so this is the unique such curve. □

Proposition 11. $\langle pt, pt \rangle_{3H-E_1-\cdots-E_6} = 12$.

Proof. Arguing as in the two previous proofs, this counts the number of degree 3 rational curves in $\mathbb{P}^2$ through 8 generic points, which is 12. This is the degree 3 case of Kontsevich's Theorem. □

Remark 12. Deland observes that each such curve is singular, with the singularity away from the 8 specified points (in particular, away from all the P_i), so it remains singular in the blowup S_3, hence each of these 12 curves will be a singular degree 3 rational curve.

Proposition 13. *For distict* $i, j \in \{1, \ldots, 6\}$,

$$\langle pt, pt \rangle_{3H-E_1-\cdots-E_6-E_i+E_j} = 1.$$

Proof. (Deland) Arguing as in the two previous proofs, the invariant counts the number of cubic curves in $\mathbb{P}^2$ passing through $\{P, Q, P_1, \ldots, P_6\} \setminus \{P_j\}$ with a singularity at P_i. (Since S_3 is obtained by blowing up the P_i, the proper transform of such a curve in S_3 is then a smooth rational curve.) The space of cubic curves in $\mathbb{P}^2$ is

$$\mathbb{P}H^0(\mathbb{P}^2, \mathscr{O}(3)) \cong \mathbb{P}^9.$$

We claim that having a singularity at a specified point imposes 3 linear conditions. Indeed, if

$$F = a_1x^3 + a_2y^3 + a_3z^3 + a_4x^2y + a_5x^2z + a_6xy^2 + a_7y^2z + a_8xz^2 + a_9yz^2 + a_{10}xyz$$

is the equation of a typical cubic, then if we require $Z(F)$ to contain, say, $[1:0:0]$ as a singular point, this imposes the conditions

$$\begin{array}{rcccl} 0 & = & F(1,0,0) & = & a_1 \\ 0 & = & F_x(1,0,0) & = & 3a_1 \\ 0 & = & F_y(1,0,0) & = & a_4 \\ 0 & = & F_z(1,0,0) & = & a_5. \end{array}$$

Requiring the curve to pass through each of the other points is a linear condition, so we impose a total of 9 linear conditions on the space of cubics, hence the number of such curves is 1. □

Having given classical geometry proofs, we will now reprove these results using the GW machinery.

Theorem 14. *Let $\beta \in A_1(S_3)$ be a degree three class, and let $\delta, \gamma \in A_1(S_3)$ be two classes with $\delta \cdot \gamma = 0$. Then we have*

$$\begin{aligned} (\gamma\cdot\gamma)(\delta\cdot\beta)\langle pt, pt\rangle_\beta &= \sum_{C(M)+M'=\beta} (C(M)\cdot M')(\gamma\cdot C(M))(\delta\cdot M')(\gamma\cdot M') \\ &\quad - \sum_{C(M)+M'=\beta} (C(M)\cdot M')(\delta\cdot C(M))(\gamma\cdot M')^2 \end{aligned}$$

Here M and M' run over all of the 27 line classes, and $C(M)$ denotes the "residual conic" class from the proof of Lemma 5.

Proof. Apply WDVV using the rational equivalence

$$D(pt, \delta|\gamma, \gamma) \sim D(pt, \gamma|\delta, \gamma)$$

in degree β. Integrating over $D(pt, \delta|\gamma, \gamma)$ gives

$$\begin{aligned} & \langle pt, \delta, pt\rangle_\beta \langle \gamma, \gamma, 1\rangle_0 \\ & + \sum_{C(M)+M'=\beta} \sum_{a=1}^{7} \langle pt, \delta, T^a\rangle_{C(M)} \langle \gamma, \gamma, T_a\rangle_{M'} \\ = \; & (\gamma\cdot\gamma)(\delta\cdot\beta)\langle pt, pt\rangle_\beta \\ & + \sum_{C(M)+M'=\beta} (C(M)\cdot M')(\delta\cdot C(M))(\gamma\cdot M')^2 \end{aligned}$$

The component with the pt, δ markings must map with degree at least two, else the invariant vanishes for dimension reasons, so this cuts down on the number of terms. The possible splittings of β are also cut down because we know a degree 2 invariant can only be nonzero if it is of class $C(M)$ for some line M and a degree 1 invariant can only be nonzero in a line class. We use Lemma 8, the Divisor Axiom, and the fact that $\langle\rangle_{M'} = 1$ and $\langle pt\rangle_{C(M)} = 1$ for the second equality.

Integrating over $D(pt, \gamma|\delta, \gamma)$ gives

$$\begin{aligned} & \langle pt, \gamma, pt\rangle_\beta \langle \delta, \gamma, 1\rangle_0 \\ & - \sum_{C(M)+M'=\beta} \sum_{a=1}^{7} \langle pt, \gamma, T^a\rangle_{C(M)} \langle \delta, \gamma, T_a\rangle_{M'} \\ = & \sum_{C(M)+M'=\beta} (C(M)\cdot M')(\gamma\cdot C(M))(\delta\cdot M')(\gamma\cdot M'). \end{aligned}$$

Note that the first term vanishes since $\delta \cdot \gamma = 0$ by hypothesis. We get the second equality using Lemma 8, the Divisor Axiom, and the fact that $\langle\rangle_{M'} = 1$ and $\langle pt\rangle_{C(M)} = 1$.

We can now equate the two expressions, and solve for $\langle pt, pt\rangle_\beta$ to complete the proof. □

Corollary 15. *The degree 3 invariants $\langle pt, pt\rangle_\beta$ are determined by WDVV, the Divisor Axiom, and the metric (the intersection form).*

Proof. It is always possible to choose δ, γ satisfying the hypotheses of Theorem 14 and so that $\gamma \cdot \gamma \neq 0$ and $\delta \cdot \beta \neq 0$. □

Corollary 16. *The degree 3 invariant $\langle pt, pt\rangle_\beta$ vanishes unless β can be written $\beta = C(M) + M'$ for line classes M, M'.*

Second Proof of Proposition 9. Apply Theorem 14 with $\beta = H$, $\delta = H$, and $\gamma = E_1$. Note $H \cdot E_1 = 0$ so the hypotheses are satisfied. There are exactly six ways to write H as $C(M) + M'$, namely: $H = C(C_i) + E_i = H - E_i + E_i$ for $i = 1, \ldots, 6$. Note that $C(C_i) \cdot E_i = 1$. We find:

$$\begin{aligned} (E_1\cdot E_1)(H.H)\langle pt, pt\rangle_H &= \sum_{i=1}^{6} (C(C_i)\cdot E_i)(E_1\cdot C(C_i))(H\cdot E_i)(E_1\cdot E_i) \\ &\quad - \sum_{i=1}^{6} (C(C_i)\cdot E_i)(H\cdot C(C_i))(E_1\cdot E_i)^2. \end{aligned}$$

All the terms in the first sum vanish because $H \cdot E_i = 0$, and all but the $i = 1$ term vanish in the second sum since $E_1 \cdot E_i = 0$ otherwise. We find

$$(-1)(1)\langle pt, pt\rangle_H = -(1)(1)(-1)^2,$$

which proves $\langle pt, pt\rangle_H = 1$.

Second Proof of Proposition 10. Again, notice that there are exactly six ways to write $2H - E_i - E_j - E_k$ as $C(M) + M'$, namely:

$$\begin{aligned} 2H - E_i - E_j - E_k &= C(C_i) + L_{jk} \\ &= C(C_j) + L_{ik} \\ &= C(C_k) + L_{ij} \\ &= C(L_{mn}) + E_l \\ &= C(L_{ln}) + E_m \\ &= C(L_{lm}) + E_n, \end{aligned}$$

where

$$\{l, m, n\} = \{1, \ldots, 6\} \setminus \{i, j, k\}.$$

Again, we also have $C(M) \cdot M' = 1$ for each of these splittings. Apply Theorem 14 with

$$\begin{aligned} \beta &= 2H - E_i - E_j - E_k \\ \delta &= E_i \\ \gamma &= E_l. \end{aligned}$$

Again, all the terms in the first sum vanish because each M' is either disjoint from E_i or disjoint from E_l, and almost all the terms in the second sum vanish because E_l is disjoint from all the M' except when $M' = E_l$. We find

$$\begin{aligned} (-1)(1)\langle pt, pt\rangle_{2H-E_i-E_j-E_k} &= -(E_i \cdot C(L_{mn}))(E_l \cdot E_l)^2 \\ &= -(1)(-1)^2, \end{aligned}$$

so $\langle pt, pt\rangle_{2H-E_i-E_j-E_k} = 1$ as desired.

Second Proof of Proposition 11: $\langle pt, pt\rangle_{3H-E_1-\cdots-E_6} = 1$. This calculation does not follow the pattern of the previous two. First of all, there are exactly 27 (!) ways to write $3H - E_1 - \cdots - E_6$ as $C(M) + M'$, namely the six expressions $C(E_i) + E_i$, the six expressions $C(C_i) + C_i$, and the 15 expressions $C(L_{ij}) + L_{ij}$. That is, there is one such expression for each line M obtained by setting $M' = M$ (c.f. Lemma 5). We have $C(M) \cdot M = 2$ for every line class M (every line meets the other two edges of a triangle containing it!). Apply Theorem 14 with

$$\begin{aligned} \beta &= 3H - E_1 - \cdots - E_6 \\ \delta &= E_1 \\ \gamma &= E_2. \end{aligned}$$

Note $E_1 \cdot E_2 = 0$ so the hypotheses are satisifed. The terms in the first sum vanish unless M' is one of the five lines adjacent to both E_1 and E_2, namely $L_{12}, C_3, C_4, C_5, C_6$, and the terms in the second sum vanish unless M' is one of the ten lines meeting E_2 or $M' = E_2$.

Since $\delta \cdot \beta = 1$ and $\gamma \cdot \gamma = -1$, we calculate

$$\begin{aligned}
-\langle pt, pt\rangle_\beta &= 2(E_2 \cdot C(L_{12})) \\
&\quad +2(E_2 \cdot C(C_3)) \\
&\quad +2(E_2 \cdot C(C_4)) \\
&\quad +2(E_2 \cdot C(C_5)) \\
&\quad +2(E_2 \cdot C(C_6)) \\
&\quad -2(E_1 \cdot C(L_{12})) \\
&\quad -2(E_1 \cdot C(L_{23})) \\
&\quad -2(E_1 \cdot C(L_{24})) \\
&\quad -2(E_1 \cdot C(L_{25})) \\
&\quad -2(E_1 \cdot C(L_{26})) \\
&\quad -2(E_1 \cdot C(C_1)) \\
&\quad -2(E_1 \cdot C(C_3)) \\
&\quad -2(E_1 \cdot C(C_4)) \\
&\quad -2(E_1 \cdot C(C_5)) \\
&\quad -2(E_1 \cdot C(C_6)) \\
&\quad -2(E_1 \cdot C(E_2)) \\
&= 0+0+0+0+0+0 \\
&\quad -2-2-2-2-2+0+0+0+0-2 \\
&= -12.
\end{aligned}$$

Multiplying through by -1 gives the result. There is nothing special about E_1 and E_2. Any two disjoint lines would work just as well, I just figured it would be best to be concrete.

Second Proof of Proposition 13. This follows the general pattern. There are six ways to write our degree 3 curve class as $C(M) + M'$, namely:

$$\begin{aligned}
3H - E_1 - \cdots - E_6 - E_i + E_j &= C(E_i) + E_j \\
&= C(L_{jk}) + L_{ik}, \qquad k \neq i, j \\
&= C(C_i) + C_j.
\end{aligned}$$

Apply Theorem 14 with

$$\begin{aligned}
\beta &= 3H - E_1 - \cdots - E_6 - E_i + E_j \\
\delta &= E_i \\
\gamma &= E_j.
\end{aligned}$$

Note $\delta \cdot \gamma = 0$, so the hypotheses are satisfied. All the terms in the first sum vanish, since, of the classes E_j, L_{ik}, C_j, only the first has nonzero intersection with $\gamma = E_j$, and it has zero intersection with $\delta = E_i$. In the second sum, only the term with $M' = E_j$ survives, and we have

$$(E_j \cdot E_j)(E_i \cdot \beta)\langle pt, pt\rangle_\beta = -(C(E_i) \cdot E_j)(E_i \cdot C(E_i))(E_j \cdot E_j)^2,$$

which simplifies to

$$(-1)(2)\langle pt, pt\rangle_\beta = -(1)(2)(-1)^2,$$

so we conclude $\langle pt, pt\rangle_\beta = 1$ as desired.

Now we can forge ahead:

Proposition 17. *For distinct $i, j, k \in \{1, \ldots, 6\}$ we have*

$$\langle pt, pt\rangle_{4H-2E_i-2E_j-2E_k-E_l-E_m-En} = 1,$$

where $\{l, m, n\} = \{1, \ldots, 6\} \setminus \{i, j, k\}$ as usual.

Proof. There are six ways to write $4H - 2E_i - 2E_j - 2E_k - E_l - E_m - E_n$ as $C(M) + M'$, namely:

$$\begin{aligned} 4H - 2E_i - 2E_j - 2E_k - E_l - E_m - E_n &= C(L_{mn}) + C_l \\ &= C(L_{ln}) + C_m \\ &= C(L_{lm}) + C_n \\ &= C(E_i) + L_{jk} \\ &= C(E_j) + L_{ik} \\ &= C(E_k) + L_{ij}. \end{aligned}$$

As usual, we have $C(M) \cdot M' = 1$ for each such splitting. Apply Theorem 14 with

$$\begin{aligned} \beta &= 4H - 2E_i - 2E_j - 2E_k - E_l - E_m - E_n \\ \delta &= E_l \\ \gamma &= L_{ij}. \end{aligned}$$

Note $\delta \cdot \gamma = 0$ so the hypotheses are satisfied. Of the lines M' in the above list, δ is disjoint from all but C_m and C_n and γ is disjoint from all but itself L_{ij}, so the terms in the first sum vanish, all the terms in the second sum vanish except the one with $M' = L_{ij}$. We find

$$\begin{aligned} -\langle pt, pt\rangle_\beta &= -(E_l \cdot C(E_k))(L_{ij} \cdot L_{ij})^2 \\ &= -(1)(-1)^2, \end{aligned}$$

from which the result follows by multiplying through by -1. □

Proposition 18. $\langle pt, pt\rangle_{5H-2E_1-\cdots-2E_6} = 1$.

Proof. There are six ways to write this class as $C(M) + M'$, namely the six $C(E_i) + C_i$. We have $C(E_i) \cdot C_i = 1$ for each i. Apply Theorem 14 with

$$\begin{aligned} \beta &= 5H - 2E_1 \cdots - 2E_6 \\ \delta &= L_{12} \\ \gamma &= C_3. \end{aligned}$$

Note $L_{12} \cdot C_3 = 0$, so the hypotheses are satisfied. Of the lines C_i, δ has nonzero intersection only with C_1 and C_2 and of course γ only meets C_3, so all the terms in the first sum vanish, and all the terms in the second sum vanish except the one where $M' = C_3$. We find

$$\begin{aligned} -\langle pt, pt\rangle_\beta &= -(L_{12} \cdot C(E_3))(C_3 \cdot C_3)^2 \\ &= -(3H - E_1 - E_2 - 2E_3 - E_4 - E_5 - E_6)(H - E_1 - E_2) \\ &= -1. \end{aligned}$$

□

Remark 19. There should be a classical proofs of Propositions 17, and 18, relying, respectively on the assertions:

1. For any eight points $P, Q, P_1, \dots, P_6 \in \mathbb{P}^2$ in general position and any distinct $i, j, k \in \{1, \dots, 6\}$, there is a unique degree 4 rational curve passing through all eight points, and passing through P_i, P_j, P_k twice.
2. For any eight points $P, Q, P_1, \dots, P_6 \in \mathbb{P}^2$ in general position, there is a unique degree 5 rational curve through P, Q and passing through each P_i twice.

I do not know how to prove these statements.

We sum up the results of this section in the following

Theorem 20. *The genus zero Gromov-Witten invariants of $S_3 \subseteq \mathbb{P}^3$ for degree three curve classes are given by:*

$$\begin{aligned} \langle pt, pt\rangle_H &= 1 \\ \langle pt, pt\rangle_{2H-E_i-E_j-E_k} &= 1 \quad (i, j, k \text{ distinct}) \\ \langle pt, pt\rangle_{3H-E_1-\cdots-E_6} &= 12 \\ \langle pt, pt\rangle_{3H-E_1-\cdots-E_6-E_i+E_j} &= 1 \quad (i, j \text{ distinct}) \\ \langle pt, pt\rangle_{4H-2E_i-2E_j-2E_k-E_l-E_m-En} &= 1 \quad (i, j, k \text{ distinct}) \\ \langle pt, pt\rangle_{5H-2E_1-\cdots-2E_6} &= 1. \end{aligned}$$

The invariants in all other degree three curve classes are zero. In particular, the total number of degree 3 rational curves in S_3 passing through two general points is

$$\langle pt, pt\rangle_3 = 1 + \binom{6}{3} + 12 + \binom{6}{2} + \binom{6}{3} + 1 = 84.$$

Proof. The only part of this we haven't fully established is the vanishing statement. By Corollary 16, invariants in classes not of the form $C(M) + M'$ will vanish. Now, one has to check that, with the exception of the classes mentioned in the theorem, any class of the form $\beta = C(M) + M'$ is equal to $C(L) + L'$ only if $M = L$ and $M' = L'$. From this it is easy to find δ and γ such that $\delta \cdot \beta \neq 0$, $\gamma \cdot \gamma \neq 0$, and $\gamma \cdot M' = 0$. The vanishing then follows from Theorem 14. □

We can use these calculations to solve Alessio Corti's exercise. Below, we set

$$h := \mathscr{O}_{\mathbb{P}^3}(1)|_{S_3} = 3H - E_1 - \cdots - E_6.$$

Proposition 21. *The quantum product $*$ preserves the subring of $A^*(S_3) \otimes \mathbb{Q}[q]$ with $\mathbb{Q}[q]$ basis $1, h, h^2 = 3pt$. In this basis, quantum multiplication by h is given by the matrix below.*

$$\begin{pmatrix} 0 & 108q^2 & 756q^3 \\ 1 & 9q & 18q^2 \\ 0 & 1 & 0 \end{pmatrix}$$

(Here the quantum parameter q indexes the degree of the curve class with respect to h.)

Proof. The first thing to notice is that

$$\begin{aligned} \sum_{\text{lines } L} [L] &= 9h \\ \sum_{\text{lines } L} [C(L)] &= 18h \end{aligned}$$

in $A_1(S_3)$. By definition of the quantum product, we have

$$\begin{aligned} h * h &= h^2 + \sum_{\alpha,L} \langle h, h, T_\alpha \rangle_L q T^\alpha + \sum_L \langle h, h, pt \rangle_{C(L)} q^2 \\ &= h^2 + \sum_{\alpha,L} (T_\alpha \cdot L) q T^\alpha + 4 \sum_L \langle pt \rangle_{C(L)} q^2 \\ &= h^2 + \sum_\alpha (T_\alpha \cdot 9h) q T^\alpha + 4 \cdot 27 q^2 \\ &= h^2 + 9qh + 108q^2, \end{aligned}$$

where $T_1, \ldots, T_7$ is some basis for $A^1(S_3)$ with dual basis $T^1, \ldots, T^7$. The Divisor Axiom, and the fact that $\langle\rangle_L = 1$ for all lines L is used for the second equality. The fact mentioned above and $\langle pt \rangle_{C(L)} = 1$ are used in the third equality. For the last equality, one can compute explicitly in the basis $(T_\alpha) = (H, E_1, \ldots, E_6)$ (for example).

Next we compute, from the definition,

$$\begin{aligned} h^2 * h &= \sum_{\alpha,L} \langle h^2, h, T_\alpha \rangle_{C(L)} q^2 T^\alpha + \sum_{\text{degree } 3\ \beta} \langle h^2, h, pt \rangle_\beta q^3 \\ &= \sum_{\alpha,L} (T_\alpha \cdot C(L)) q^2 T^\alpha + 9 \sum_{\text{degree } 3\ \beta} \langle pt, pt \rangle_\beta q^3 \\ &= \sum_\alpha (T_\alpha \cdot 18h) q^2 T^\alpha + 9 \cdot 84 q^3 \\ &= 18q^2 h + 756 q^3 \end{aligned}$$

by using the Divisor Axiom, the above observation about the sum of the residual conic classes, and the degree 3 invariant computations in Theorem 20.

□

Keep on the Sunny Side,

—Gillam