MATH 251 ALGEBRA I
PROBLEMS

Numbered exercises are from Dummit and Foote.

(1) More on categories.
   (a) prove that for an object $A \in Ob(C)$ the identity $id_A \in Hom(A, A)$ is unique.
   (b) prove that the inverse of an isomorphism is unique
   (c) Consider a partially ordered set $X$ and let $Cat(X)$ be the associated categories (a unique arrow $x \to y$ for each pair $x \leq y$). Show that the product of $x$ and $y$ in $Cat(X)$, if exists, is the greatest lower bound of $x, y$. Identify similarly the coproduct.
   (d) Use the previous exercise to cook up a category where products and coproducts don’t always exist.
   (e) Let $Y$ be a set and $P(Y)$ be the set of all subsets of $Y$, partially ordered by inclusion. Identify explicitly products and coproducts in $Cat(P(Y))$.
   (f) Let $A \to B$ be an $R$-module homomorphism. What is the fibered product $A \times_B 0$ in elementary terms? What is the cofibered coproduct $0 \sqcup^A B$?
   (g) Give a canonical isomorphism $A \times_B (B \times_C D) \simeq A \times_C D$ (draw the relevant diagram! the maps are important!).
   (h) In $R$-mod, consider $D = A \times_B C$. Show if $A \to B$ injective then $D \to C$ injective, and the same for surjectivity.
   (i) Prove that a group object in $Sets$ is a group. Prove that a group object in $Groups$ is an abelian group.

(2) Localization: Section 7.5 p. 264 ex. 2,5,6. Section 15.4 p. 726 ex. 1,2,14,16 (note prop. 4.1!)

(3) Limits:
   (a) Give an example of a directed system of modules $(M_i, \phi_{ij})$ with $M_i \neq 0$ for all $i$ but $\lim_{\rightarrow} M_i = 0$.
   (b) Define a homomorphism of directed systems $h : (M_i, \phi_{ij}) \to (N_i, \psi_{ij})$ with the same index set $I$ to be a collection of homomorphisms $h_i : M_i \to N_i$ such that $h_j \circ \phi_{ij} = \psi_{ij} \circ h_i$ (draw the diagrams). Show that this is a category $R - mod_I$, and the limit $\lim_{\leftarrow} : R - mod_I \to R - mod$ is a functor.

(4) Tensor products: p. 375 ex. 2,3,4,5,6,8d,15,19